The All-Loop S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills

Jacob L. Bourjaily
Princeton University & IAS

in collaboration with

N. Arkani-Hamed, F. Cachazo, and J. Trnka

also with Andrew Hodges and S. Caron-Huot,

[arXiv:1012.6032], [arXiv:1012.6030], [arXiv:1008.2958],
([arXiv:1006:1899], [arXiv:0912.4912], [arXiv:0912.3249])

Outline

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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I = 2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathcal{D}^\dagger, \mathcal{D}_\rho^\dagger, \mathcal{D}_\sigma^\dagger, \mathcal{D}_\tau^\dagger)_{\binom{0}{2}} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathcal{D} \\ \mathcal{D}_\rho \\ \mathcal{D}_\sigma \\ \mathcal{D}_\tau \end{pmatrix}_{\binom{0}{2}}, \qquad (6)$$

where $\mathcal{D}$, $\mathcal{D}_\rho$, $\mathcal{D}_\sigma$ and $\mathcal{D}_\tau$ are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6, and K, K_ρ, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors $\mathcal{D}_\rho$, $\mathcal{D}_\sigma$ and $\mathcal{D}_\tau$ are obtained from the vector $\mathcal{D}$ by the permutations $(p_2 \leftrightarrow p_3)$, $(p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in $\mathcal{D}$. The individual components of the vector $\mathcal{D}$ represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N = 3$ for QCD):

$$K = \tfrac{1}{8} g^8 N^4 (N^2-1) K^{(4)} + \tfrac{1}{2} g^8 N^2 (N^2-1) K^{(2)}. \qquad (7)$$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector $\mathcal{D}$ is related to the thirty-three diagrams $D^G (I = 1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^F (I = 1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^S (I = 1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\mathcal{D}_0 = \frac{2 s_{14}}{\sqrt{|s_{15} s_{45} s_{16} s_{46}|}\, s_{23} s_{56}} \{ t_{123}^2 C^G \cdot D_0^G - 4 s_{14} t_{123} E(p_5 + p_6, p_6) C^F \cdot D_0^F - 2 s_{14} G(p_5 + p_6, p_5 + p_6) C^S \cdot D_0^S \},$$

$$\mathcal{D}_2 = \frac{s_{56}}{s_{23}} C^G \cdot D_2^G, \qquad (8)$$

where the constant matrices $C^G(11 \times 33)$, $C^F(11 \times 11)$ and $C^S(11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by

$$E(p_i, p_j) = \tfrac{1}{2} \{ (p_1 p_4)(p_i p_j) - (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4) + i \varepsilon_{\mu\nu\rho\lambda} p_1^\mu p_i^\nu p_j^\rho p_4^\lambda \} / (p_1 p_4),$$

$$G(p_i, p_j) = E(p_i, p_5) E(p_j, p_6), \qquad (9)$$

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TABLE 1

Matrices $K(I,J)[I=1\text{-}11,\ J=1\text{-}11]$.

Matrix $K^{(4)}$

8	4	−2	2	−1	2	0	1	0	0	−1
4	8	−1	1	−1	0	2	1	0	1	−1
−2	−1	8	4	4	1	1	2	2	1	2
2	1	4	8	2	−1	−1	4	1	1	1
−1	−1	4	2	8	1	2	4	−2	−1	4
2	0	1	−1	1	8	4	−1	0	1	0
0	2	1	−1	2	4	8	−2	0	0	0
1	1	2	4	4	−1	−2	8	−1	−1	2
0	0	2	1	−2	0	0	−1	8	4	−2
0	1	1	1	−1	1	0	−1	4	8	−1
−1	−1	2	1	4	0	0	2	−2	−1	8

Matrix $K^{(2)}$

0	0	0	0	0	0	0	0	3	3	−3
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
−3	0	0	0	0	0	0	0	0	0	0

Matrix $K_\rho^{(4)}$

0	0	0	0	1	1	0	1	1	0	−1
0	0	0	0	2	0	1	1	2	1	−2
0	0	0	0	0	1	1	1	0	1	1
0	0	0	0	1	0	0	2	0	1	0
1	2	0	1	0	1	2	2	0	0	2
1	0	1	0	1	4	2	0	0	0	−1
0	1	1	0	2	2	4	0	0	0	−2
1	1	1	2	2	0	0	4	0	0	0
1	2	0	0	0	0	0	0	0	2	−1
0	1	1	1	0	0	0	0	2	4	0
−1	−2	1	0	2	−1	−2	0	−1	0	4

Matrix $K_\rho^{(2)}$

3	3	0	3	0	0	0	3	0	0	0
3	3	0	0	0	0	0	0	0	0	0
0	0	3	0	0	3	0	0	3	3	0
3	0	0	0	0	3	0	0	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	3	3	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	0	3	0
3	0	0	0	0	0	0	0	3	0	0
0	0	3	0	3	3	0	3	0	0	0
0	0	3	0	3	3	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Matrix $K_\sigma^{(4)}$

4	2	0	2	0	1	0	1	0	0	0
2	4	0	1	0	0	1	1	0	1	0
0	0	4	2	2	1	1	1	2	1	0
2	1	2	0	1	2	1	0	1	0	0
0	0	2	1	0	0	0	0	4	2	2
1	0	1	2	0	0	0	0	1	2	0
0	1	1	1	0	0	0	0	2	4	0
1	1	1	0	0	0	0	0	2	0	1
0	0	2	1	4	1	2	2	0	0	−4
0	1	1	0	2	2	4	0	0	0	−2
0	0	0	0	2	0	0	1	−4	−2	4

Matrix $K_\sigma^{(2)}$

0	0	0	0	0	0	0	0	3	0	−3
0	0	0	0	3	0	0	3	0	0	0
0	0	0	0	0	0	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	3	0	0	0	0
0	0	0	0	0	3	3	0	0	0	−3
0	0	3	0	3	3	3	0	0	0	0
0	3	0	0	0	0	0	3	0	0	0
3	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	3	0
−3	0	0	0	0	−3	0	0	0	0	0

Matrix $K_\tau^{(4)}$

0	1	−1	−1	1	1	0	1	2	0	0
1	0	−2	−1	2	0	1	1	4	2	0
−1	−2	0	0	0	1	1	1	−1	1	0
−1	−1	0	1	0	2	1	0	1	−1	0
1	2	0	0	1	−1	−1	0	−2	2	1
1	0	1	2	−1	0	1	−2	2	4	−1
0	1	1	1	−1	1	0	−1	4	8	−1
1	1	1	0	0	−2	−1	0	2	−2	0
2	4	−1	1	−2	2	4	2	1	0	−2
0	2	1	−1	2	4	8	−2	0	0	0
0	0	0	0	1	−1	−1	0	−2	0	2

Matrix $K_\tau^{(2)}$

3	3	0	0	0	3	3	0	0	0	0
3	3	0	0	0	3	3	0	0	0	0
0	0	3	3	3	0	0	3	0	0	0
0	0	3	3	3	0	0	3	0	0	3
0	0	3	3	3	0	0	3	0	0	0
3	3	0	0	0	3	3	0	0	0	−3
3	3	0	0	0	3	3	0	0	0	0
0	0	3	3	3	0	0	3	0	0	0
0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	0	0	0	0	3	3	0
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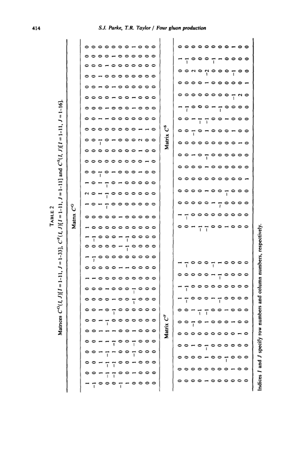

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where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt} = 1$. For the future use, we define one more function,

$$F(p_i, p_j) = \{(p_1 p_4)(p_i p_j) + (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4)\}/(p_1 p_4) . \qquad (10)$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6.

The diagrams D_2^G are listed below:

$$D_2^G(1) = \frac{\delta_2}{s_{14}s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)] - [(p_2-p_5)(p_3+p_6)] \times [(p_1-p_4)(p_3-p_6)] + [(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\} ,$$

$$D_2^G(2) = \frac{1}{s_{25}s_{36}}\{2E(p_2-p_5, p_3-p_6) - 2E(p_3-p_6, p_2-p_5) + \delta_2[(p_2-p_5)(p_3-p_6)]\} ,$$

$$\begin{aligned} D_2^G(3) = \frac{4}{s_{25}s_{36}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6) \\ &-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3) \\ &+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5, p_6) \\ &-[p_1(p_2-p_5)]E(p_3-p_6, p_3+p_6) - [p_4(p_3-p_6)]E(p_2+p_5, p_2-p_5) \\ &+\delta_2[p_1(p_2-p_5)][p_4(p_3-p_6)]\} , \end{aligned}$$

$$D_2^G(4) = \frac{-2}{s_{36}t_{125}}\{E(p_3-p_6, p_3+p_6) - \delta_2[p_4(p_3-p_6)]\} ,$$

$$D_2^G(5) = \frac{-2}{s_{25}t_{125}}\{E(p_2+p_5, p_2-p_5) - \delta_2[p_1(p_2-p_5)]\} ,$$

$$D_2^G(6) = \frac{\delta_2}{t_{125}} ,$$

$$\begin{aligned} D_2^G(7) = \frac{4}{s_{12}s_{36}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6) - [p_4(p_3-p_6)]E(p_2, p_2-p_5)\} , \end{aligned}$$

$$\begin{aligned} D_2^G(8) = \frac{4}{s_{34}s_{25}t_{125}}\{&[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3) \\ &-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3) - [p_1(p_2-p_5)]E(p_3-p_6, p_3)\} , \end{aligned}$$

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$$D_2^G(9) = \frac{4}{s_{15}s_{36}t_{125}}\{[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)$$
$$-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_4(p_3-p_6)]E(p_5,p_2-p_5)\}\,,$$

$$D_2^G(10) = \frac{4}{s_{25}s_{46}t_{125}}\{[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)$$
$$-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_1(p_2-p_5)]E(p_3-p_6,p_6)\}\,,$$

$$D_2^G(11) = \frac{\delta_2}{s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(12) = \frac{-\delta_2}{s_{36}t_{145}}[s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}}[s_{12}-s_{24}][s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}}[s_{15}-s_{45}][s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(15) = \frac{\delta_2}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_2^G(16) = \frac{-4}{s_{12}s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]E(p_2,p_2)\,,$$

$$D_2^G(17) = \frac{4}{s_{36}s_{45}t_{145}}[s_{23}-s_{26}-s_{36}]E(p_5,p_5)\,,$$

$$D_2^G(18) = \frac{-4}{s_{12}s_{36}s_{45}}[2(p_1+p_2)(p_3-p_6)-s_{36}]E(p_2,p_5)\,,$$

$$D_2^G(19) = \frac{-2}{s_{12}s_{36}}E(p_2,p_3-p_6)\,,$$

$$D_2^G(20) = \frac{2}{s_{36}s_{45}}E(p_3-p_6,p_5)\,,$$

$$D_2^G(21) = \frac{-4}{s_{25}s_{34}t_{134}}[s_{26}-s_{56}+s_{25}]E(p_3,p_3)\,,$$

$$D_2^G(22) = \frac{4}{s_{16}s_{25}t_{146}}[s_{23}-s_{35}-s_{25}]E(p_6,p_6)\,,$$

$$D_2^G(23) = \frac{4}{s_{16}s_{25}s_{34}}[2(p_1+p_6)(p_2-p_5)+s_{25}]E(p_6,p_3)\,,$$

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$$D_2^{G}(24) = \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3)\,,$$

$$D_2^{G}(25) = \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5)\,,$$

$$D_2^{G}(26) = \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5)\,,$$

$$D_2^{G}(27) = \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6)\,,$$

$$D_2^{G}(28) = \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5)\,,$$

$$D_2^{G}(29) = \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3)\,,$$

$$D_2^{G}(30) = \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125}] E(p_2, p_3)\,,$$

$$D_2^{G}(31) = \frac{4}{s_{12}s_{46}t_{125}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}] E(p_2, p_6)\,,$$

$$D_2^{G}(32) = \frac{4}{s_{15}s_{34}t_{125}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3)\,,$$

$$D_2^{G}(33) = \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125}] E(p_5, p_6)\,, \qquad (11)$$

where $\delta_2 = 1$.

The diagrams D_0^{G} are obtained from D_2^{G} by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams D_0^{F} are listed below:

$$D_0^{F}(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) + [F(p_6, p_3) + s_{34}]E(p_5, p_5)\}\,,$$

$$D_0^{F}(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{[F(p_6, p_2) + \tfrac{1}{2}s_{16}]E(p_3, p_5) + [F(p_2, p_3) + \tfrac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)\}\,,$$

$$D_0^{F}(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) - [F(p_3, p_6) - \tfrac{1}{2}s_{36} - \tfrac{1}{2}s_{34} + \tfrac{1}{2}s_{46}]E(p_5, p_5)\}\,,$$

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$$D_0^{\rm F}(4) = \frac{4}{s_{25}s_{34}t_{125}}\{F(p_2,p_3)E(p_5,p_5) - F(p_5,p_3)E(p_2,p_5)$$
$$+[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)\}\,,$$

$$D_0^{\rm F}(5) = \frac{2}{s_{16}s_{25}t_{146}}[s_{35} - s_{23} + s_{25}]E(p_6,p_5)\,,$$

$$D_0^{\rm F}(6) = \frac{2}{s_{25}s_{34}t_{134}}[s_{56} - s_{26} - s_{25}]E(p_3,p_5)\,,$$

$$D_0^{\rm F}(7) = \frac{4}{s_{25}s_{36}t_{125}}\{[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)$$
$$+[F(p_2,p_3) + \tfrac{1}{4}t_{125}]E(p_5,p_5) - [F(p_5,p_3) + \tfrac{1}{4}t_{125}]E(p_2,p_5)\}\,,$$

$$D_0^{\rm F}(8) = \frac{1}{s_{14}s_{36}}E(p_3 - p_6, p_5)\,,$$

$$D_0^{\rm F}(9) = \frac{2}{s_{14}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2,p_5)\,,$$

$$D_0^{\rm F}(10) = \frac{2}{s_{14}s_{36}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5,p_5)\,,$$

$$D_0^{\rm F}(11) = \frac{1}{2s_{14}s_{25}s_{36}}\{[s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5)$$
$$-[s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)\}\,. \qquad (12)$$

The diagrams $D_0^{\rm S}$ are listed below:

$$D_0^{\rm S}(1) = \frac{1}{s_{25}s_{36}t_{125}}[s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}]\,,$$

$$D_0^{\rm S}(2) = \frac{1}{s_{14}s_{36}t_{124}}[s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}]\,,$$

$$D_0^{\rm S}(3) = \frac{1}{s_{14}s_{36}t_{145}}[s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}]\,,$$

$$D_0^{\rm S}(4) = \frac{1}{s_{15}s_{36}t_{125}}[s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}]\,,$$

$$D_0^{\rm S}(5) = \frac{1}{s_{15}s_{34}t_{156}}[s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}]\,,$$

$$D_0^{\rm S}(6) = \frac{1}{s_{15}s_{34}t_{125}}[s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}]\,,$$

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$$D_0^S(7) = \frac{1}{s_{25}s_{34}t_{125}}[s_{36} - s_{46} + s_{34}][s_{12} - s_{15} - s_{25}]\,,$$

$$D_0^S(8) = \frac{1}{s_{16}s_{25}t_{146}}[s_{25} + s_{35} - s_{23}][s_{14} - s_{46} + s_{16}]\,,$$

$$D_0^S(9) = \frac{1}{s_{25}s_{34}t_{134}}[s_{14} + s_{34} - s_{13}][s_{26} - s_{56} + s_{25}]\,,$$

$$D_0^S(10) = \frac{1}{s_{25}s_{36}}(p_2 - p_5)(p_3 - p_6)\,,$$

$$D_0^S(11) = \frac{1}{s_{14}s_{36}}(p_1 - p_4)(p_3 - p_6)\,,$$

$$D_0^S(12) = \frac{1}{s_{16}s_{25}}(p_6 - p_1)(p_2 - p_5)\,,$$

$$D_0^S(13) = \frac{1}{s_{15}s_{34}}(p_5 - p_1)(p_3 - p_4)\,,$$

$$D_0^S(14) = \frac{1}{s_{25}s_{34}}(p_2 - p_5)(p_3 - p_4)\,,$$

$$D_0^S(15) = \frac{1}{s_{14}s_{25}s_{36}}\{[(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)] + [(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] + [(p_1 + p_4)(p_2 - p_5)][(p_1 - p_4)(p_3 - p_6)]\}\,,$$

$$D_0^S(16) = \frac{2}{s_{16}s_{34}s_{25}}\{[(p_2 - p_5)(p_3 + p_4)][(p_1 - p_6)(p_3 - p_4)] + [(p_1 + p_6)(p_3 - p_4)][(p_1 - p_6)(p_2 - p_5)] + [(p_1 - p_6)(p_2 + p_5)][(p_3 - p_4)(p_2 - p_5)]\}\,. \quad (13)$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors $\mathcal{D}_0$ and $\mathcal{D}_2$. After generating the vectors $\mathcal{D}_{0_\rho}$, $\mathcal{D}_{0_\sigma}$, $\mathcal{D}_{0_\tau}$, $\mathcal{D}_{2_\rho}$, $\mathcal{D}_{2_\sigma}$ and $\mathcal{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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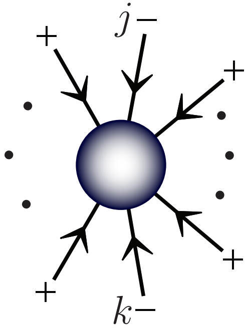

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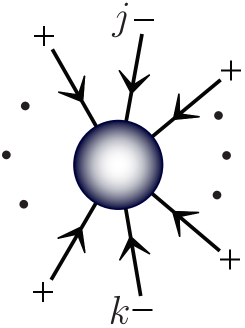

Generalizing Parke-Taylor's Formula Through 3-Loops:

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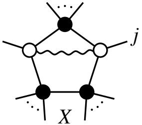

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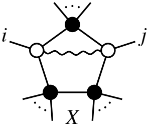

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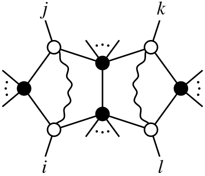

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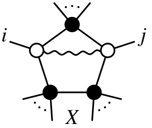

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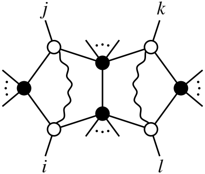
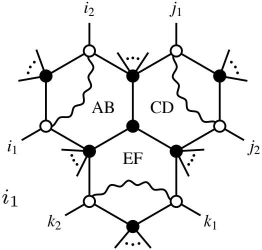


$$+\;\frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq\\ \leq j_2<k_1\leq k_2<i_1}}$$



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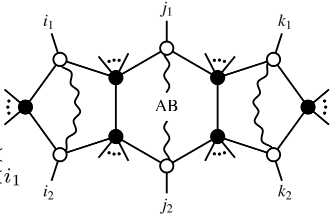

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Simple Sources of Simplification

An n-point scattering amplitude is specified by listing each particle's:

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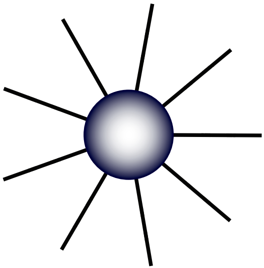

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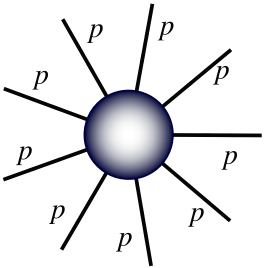

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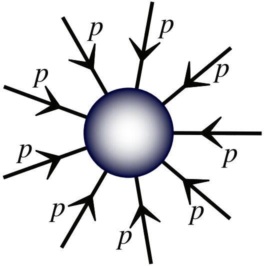

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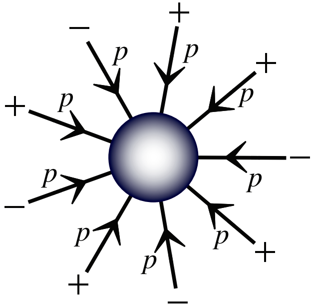

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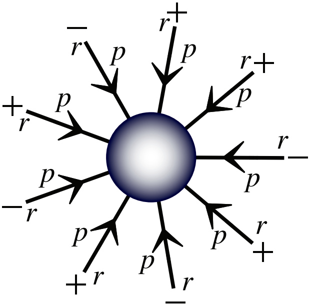

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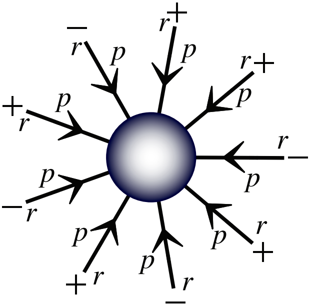

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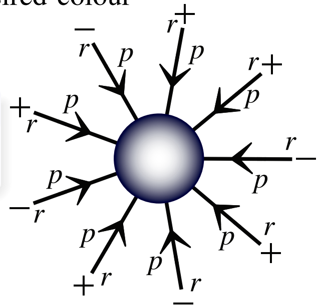
By shuffling all colour-factors to the outside of every Feynman diagram, we can write the amplitude* for any desired colour-ordering in terms of any other.



Colour-ordered partial amplitudes

$$A_n(\{p_a\}) = \sum \text{Tr}(T^{a_1}\cdots T^{a_n})\mathcal{A}_n(p_{a_1},\ldots,p_{a_n})$$

e.g. $\mathcal{A}_9(1^+,2^+,3^-,4^+,5^-,6^+,7^-,8^+,9^-)$

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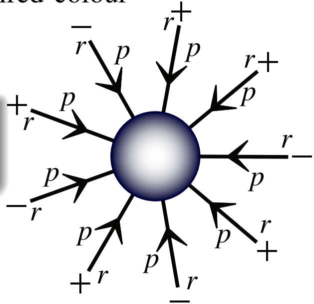

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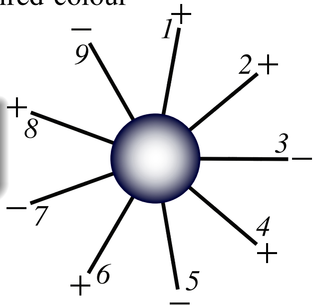

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Useful Lorentz-invariant scalars:

$$\langle ab\rangle \equiv \begin{vmatrix} \lambda_a^1 & \lambda_b^1 \\ \lambda_a^2 & \lambda_b^2 \end{vmatrix}, \qquad [ab] \equiv \begin{vmatrix} \widetilde{\lambda}_a^{\dot 1} & \widetilde{\lambda}_b^{\dot 1} \\ \widetilde{\lambda}_a^{\dot 2} & \widetilde{\lambda}_b^{\dot 2} \end{vmatrix}$$

$$(p_a + p_b)^2 = \langle ab\rangle[ba] \equiv s_{ab}, \qquad \langle a|(b + \ldots + c)|d] \equiv \langle a|\Big(b\rangle[b + \ldots + c\rangle[c\Big)|d].$$

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In $\mathcal{N} = 4$, all external states are related by supersymmetry.

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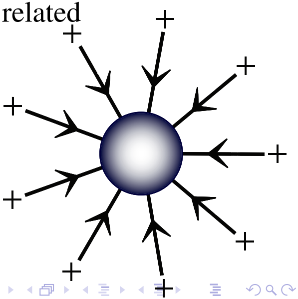
N^{k}MHV Classification of Amplitudes

- $\mathcal{A}_n^{(m=0)}(+, \ldots, +) = 0$
- $\mathcal{A}_n^{(1)}(+, \ldots, -, \ldots, +) = 0 \quad (n > 3)$
- $\mathcal{A}_n^{(2)}(j^-, \ldots, k^-) = \dfrac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$



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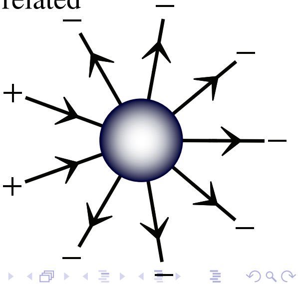

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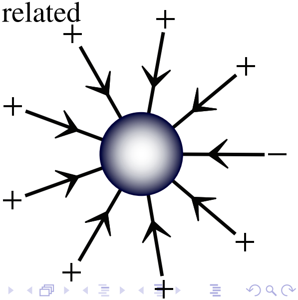

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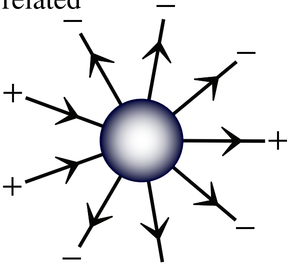

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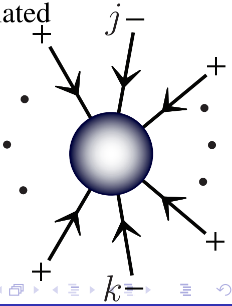

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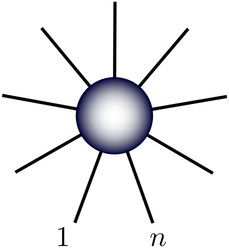
Analytic S-Matrix Redux: Tree-Level Recursion Relations

Tree amplitudes are entirely fixed by analyticity.

Consider the simplest deformation of any amplitude: $\mathcal{A}_n \mapsto \widehat{\mathcal{A}}_n(z)$

$$\lambda_1 \mapsto \widehat{\lambda}_1 \equiv \lambda_1 + z\lambda_n$$



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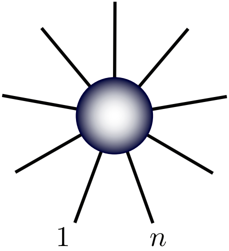

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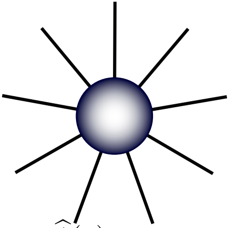

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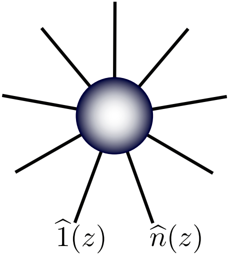

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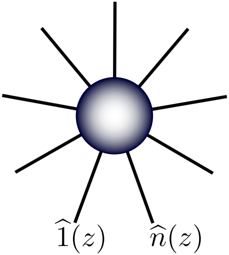

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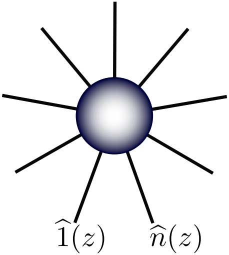

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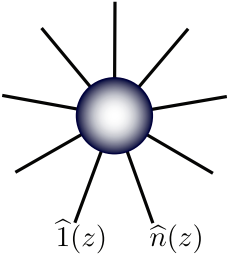

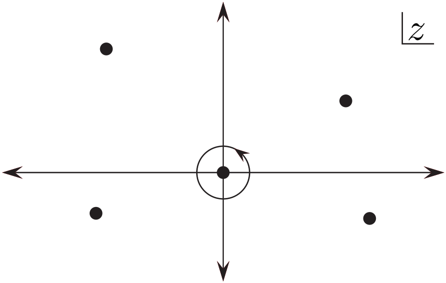

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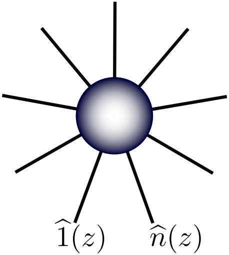

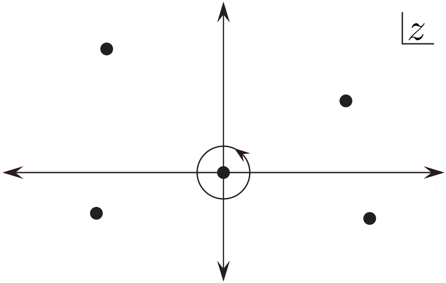

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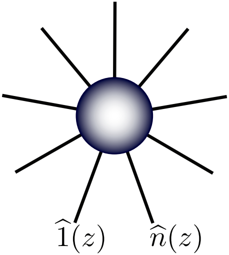

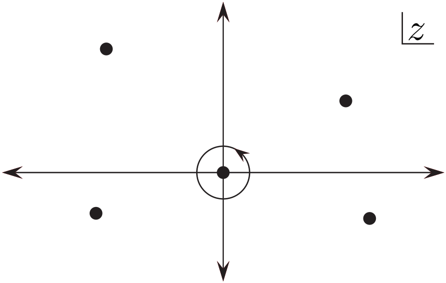

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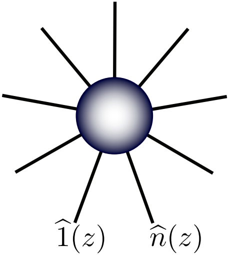

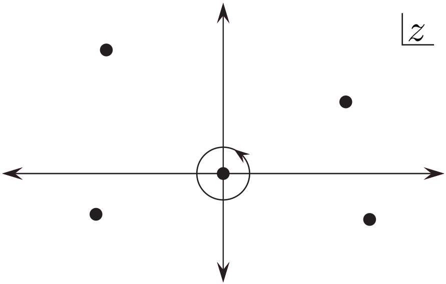

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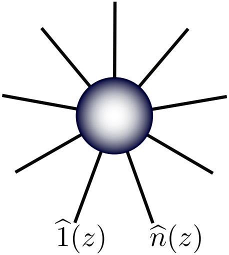

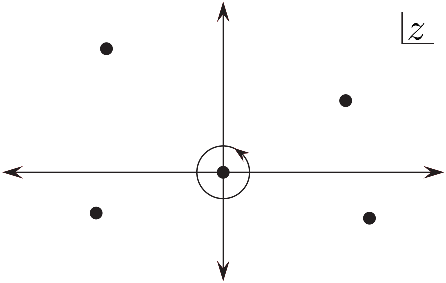

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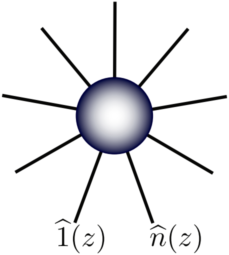

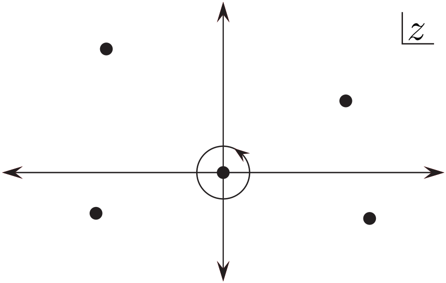

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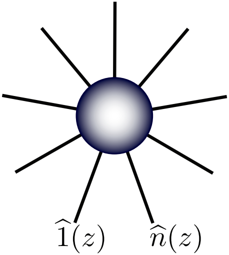

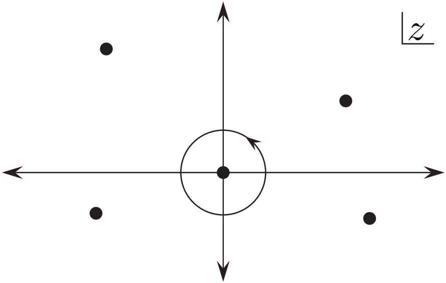

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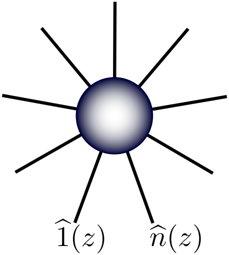

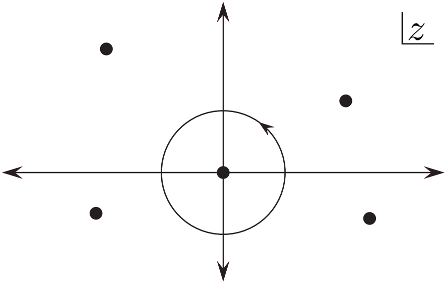

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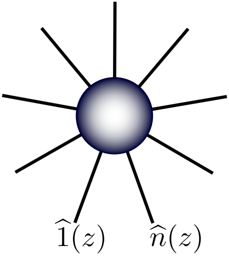

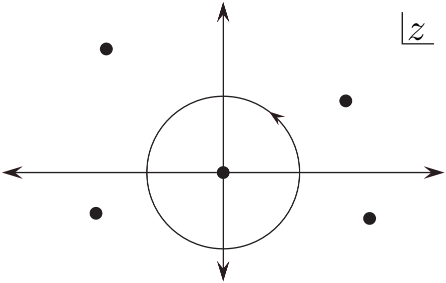

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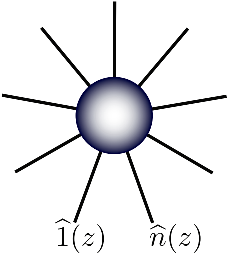

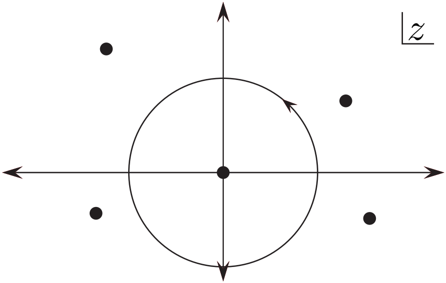

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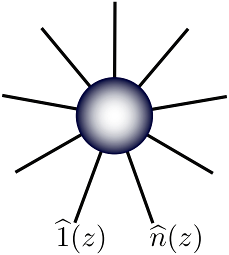

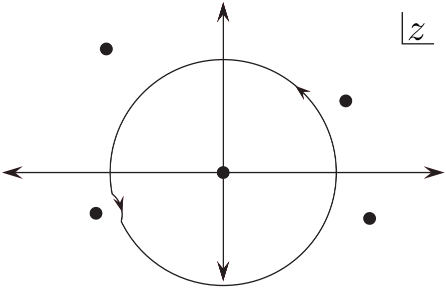

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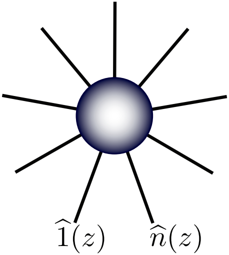

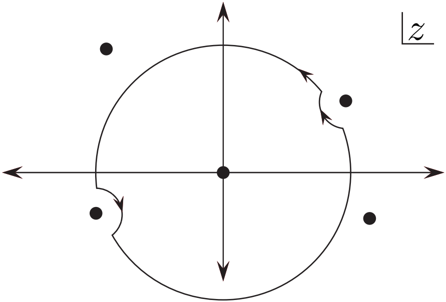

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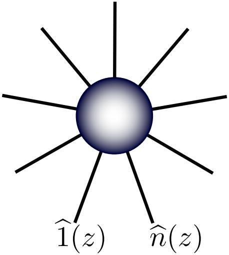

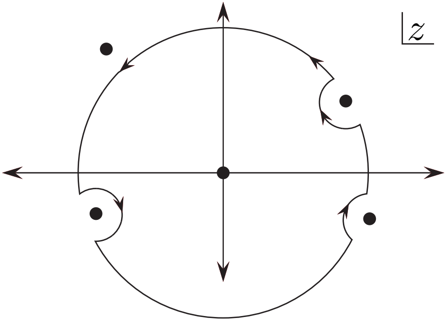

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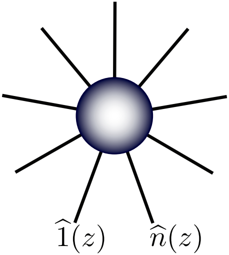

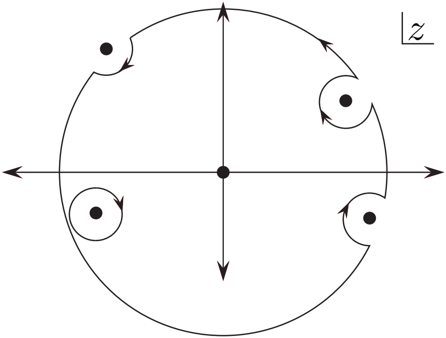

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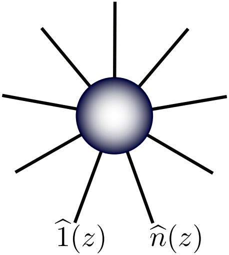

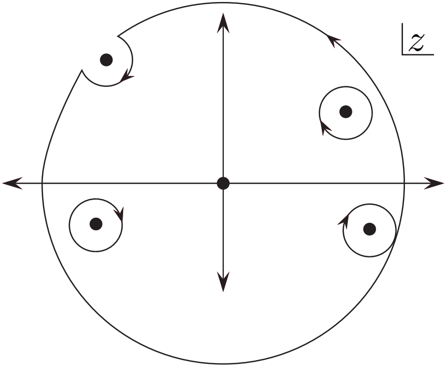

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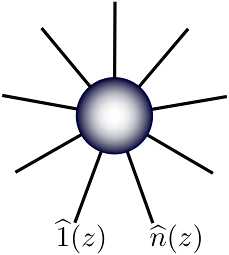

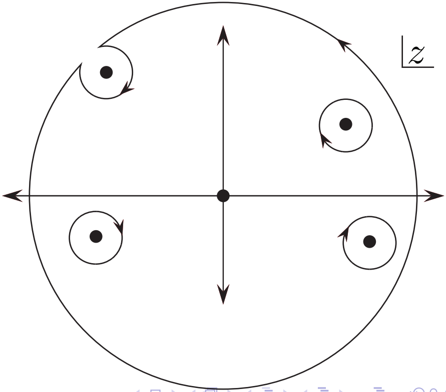

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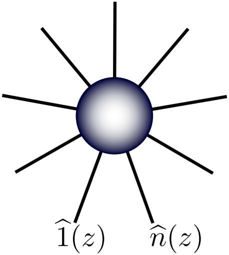

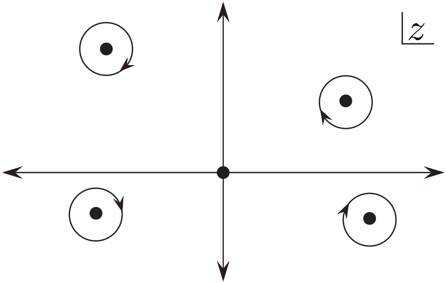

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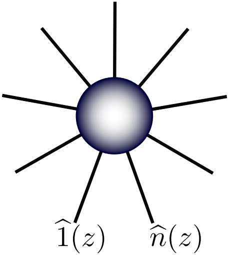

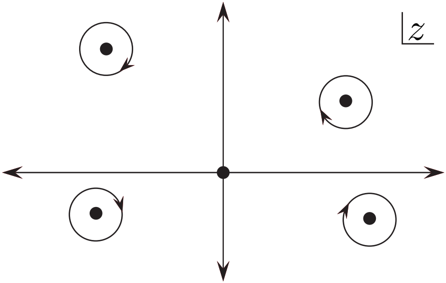

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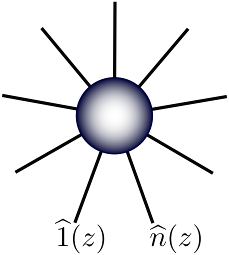

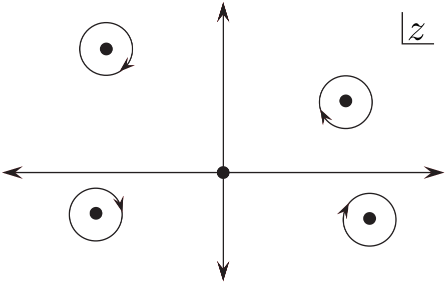

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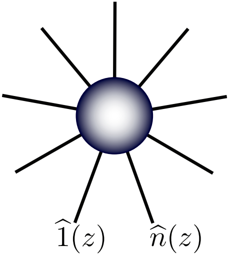

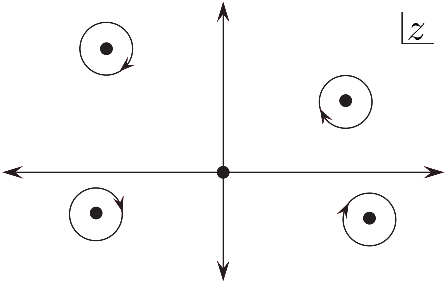

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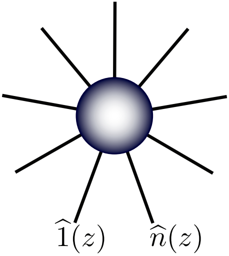

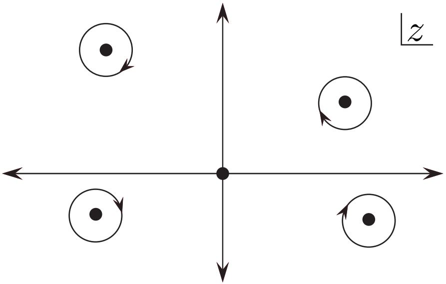

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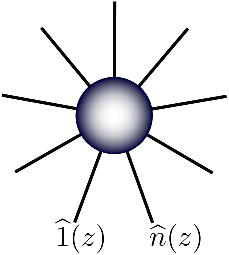

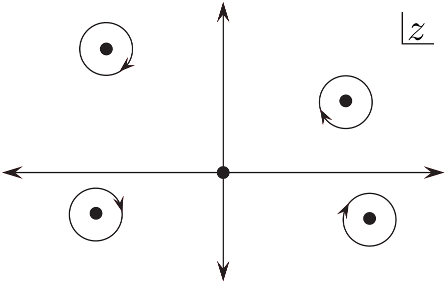

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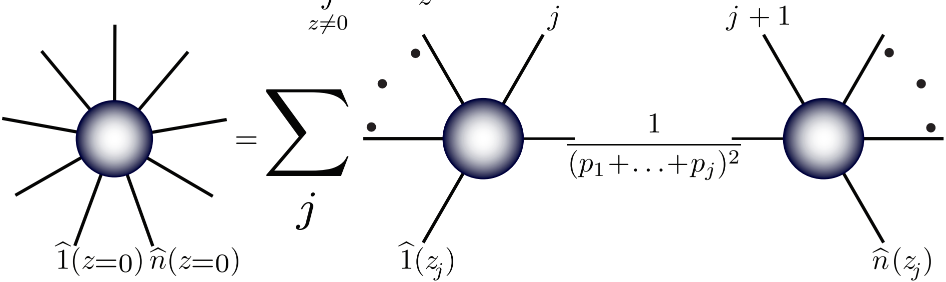

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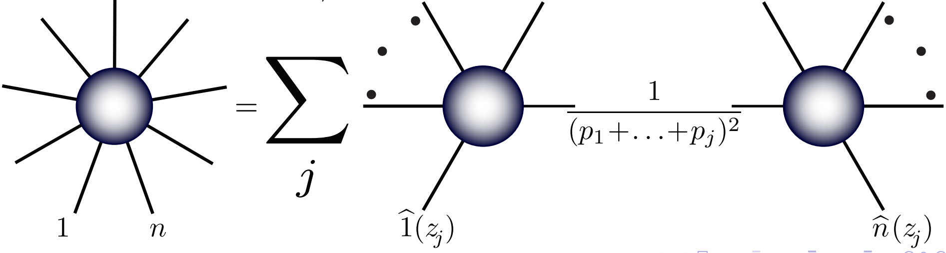

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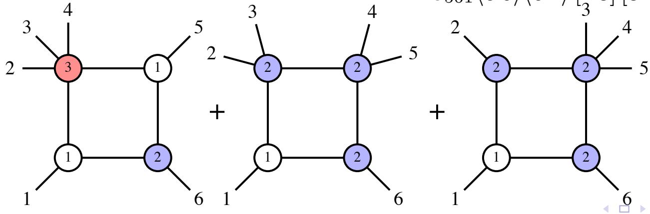

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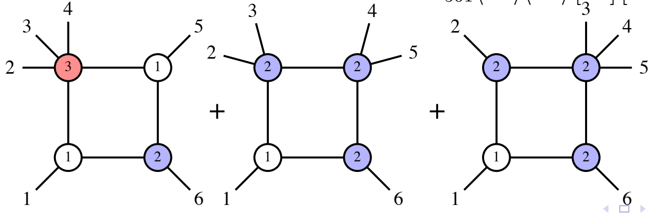

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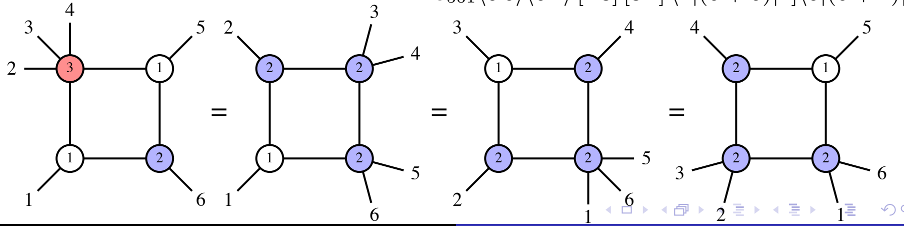

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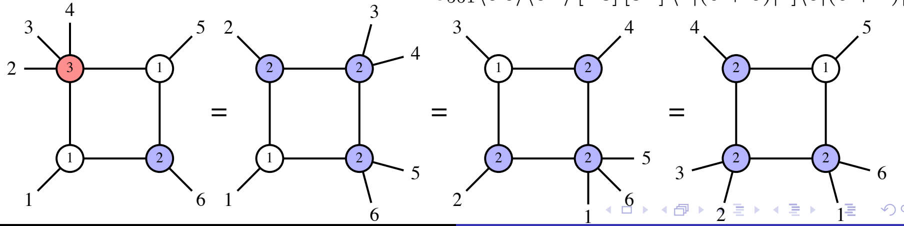

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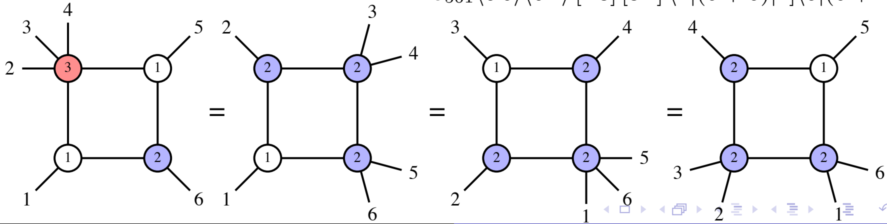

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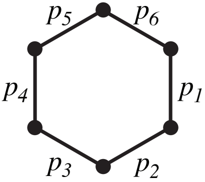

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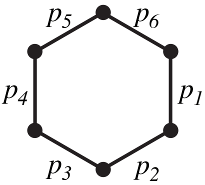
Solution: dual-coordinate x-space.

- $p_a \equiv x_{a+1} - x_a$
- scattering amplitudes turn out to be superconformal invariant with respect to these dual-coordinates!
- combined with the ordinary-space superconformal invariance, scattering amplitudes are invariant under an infinite-dimensional Yangian symmetry.



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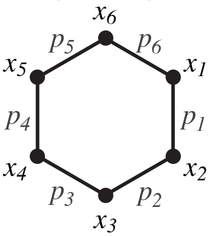

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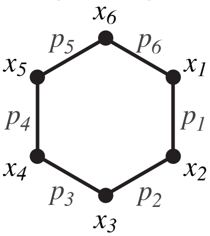

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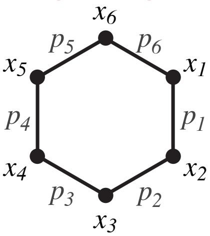

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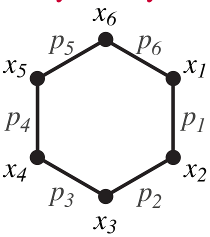

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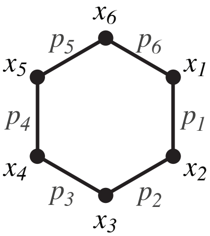

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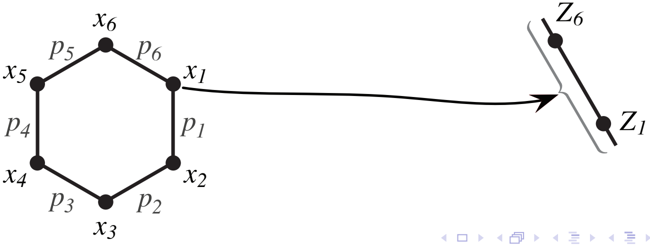

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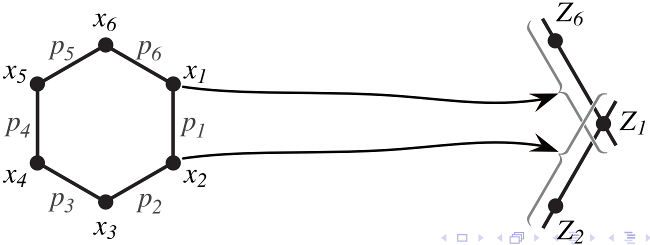

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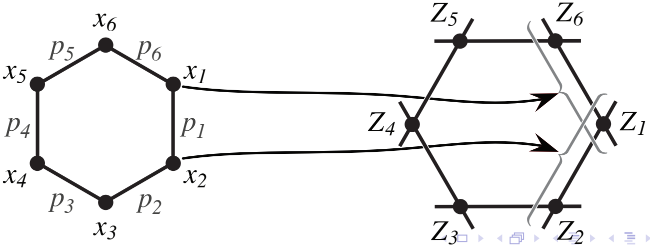

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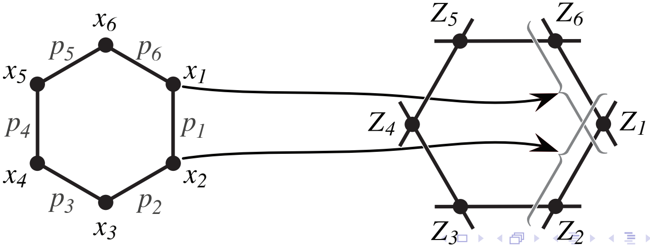

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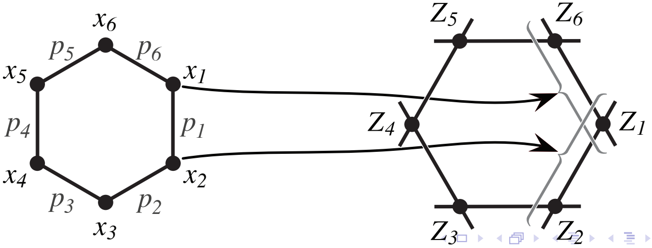

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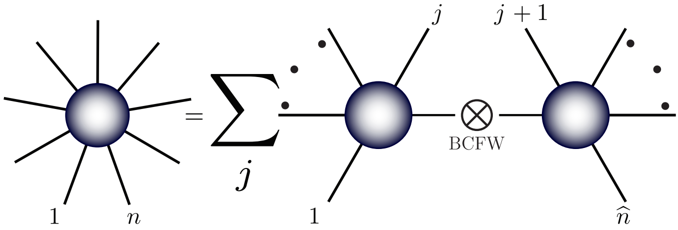

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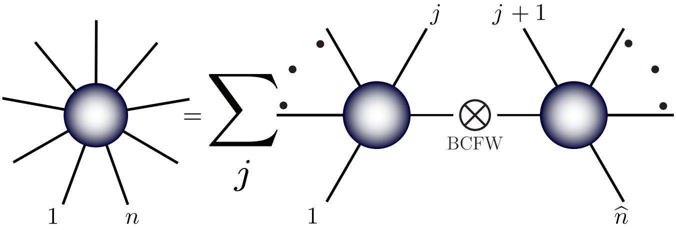

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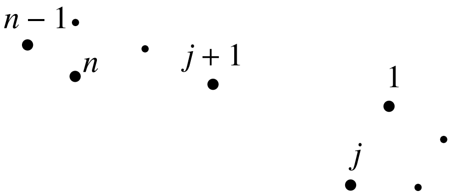

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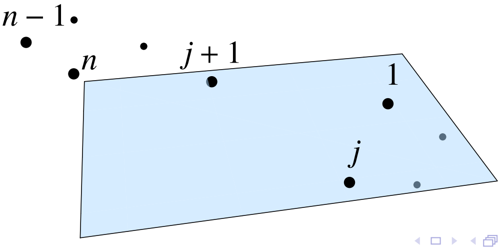

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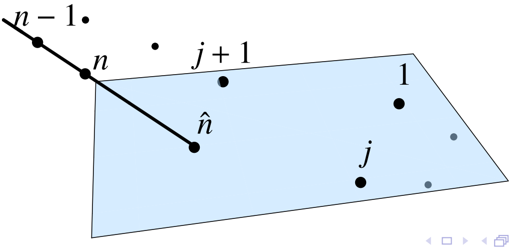

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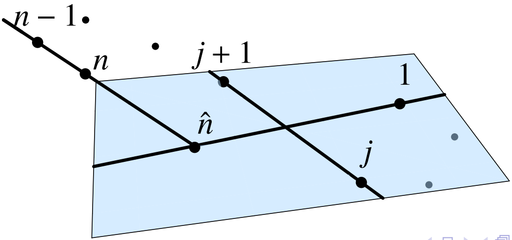

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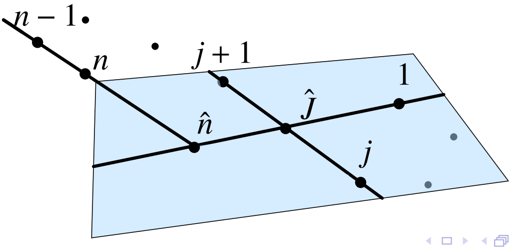

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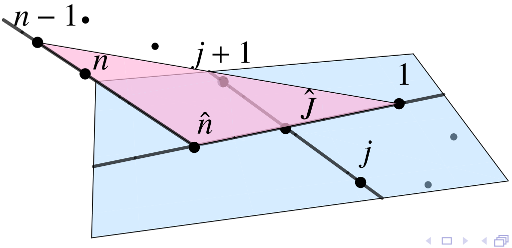

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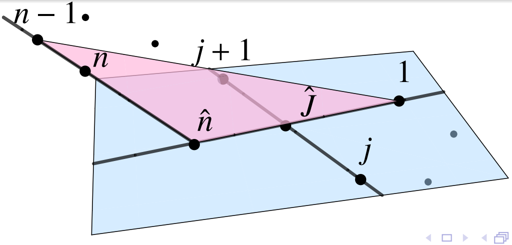

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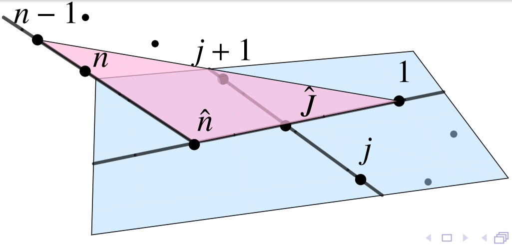

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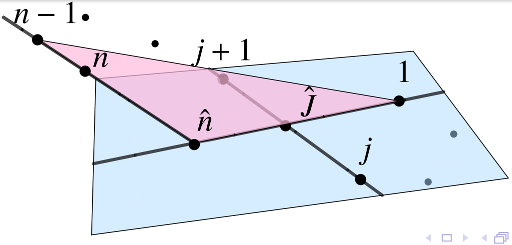

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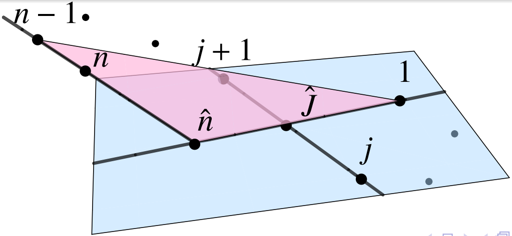

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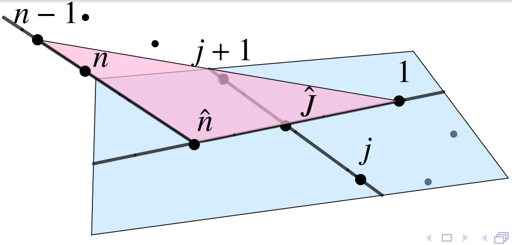
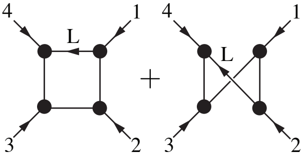


The Meaning of *The* Loop Integrand

In a general theory, there is no naturally well-defined way to combine disparate Feynman loop integrals:



$$= \begin{cases} \displaystyle\int d^4\ell_1 \frac{(p_1+p_2)^2(p_2+p_3)^2}{\ell_1^2(\ell_1-p_1)^2(\ell_1-p_1-p_2)^2(\ell_1+p_4)^2} \\ \displaystyle\int d^4\ell_2 \frac{(p_1+p_2)^2(p_2+p_3)^2}{\ell_2^2(\ell_2-p_2)^2(\ell_2-p_1-p_2)^2(\ell_2+p_4)^2} \end{cases}$$

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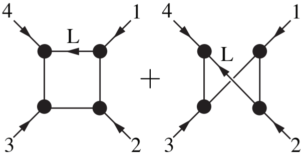

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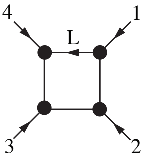

$$= \int d^4L \frac{(p_1+p_2)^2(p_2+p_3)^2}{L^2(L-p_1)^2(L-p_1-p_2)^2(L+p_4)^2}$$

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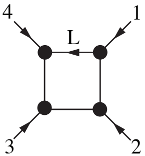

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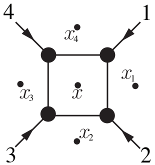

$$= \int d^4x \frac{(x_1-x_3)^2(x_2-x_4)^2}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}$$

Integrals over Lines in Momentum-Twistor Space

Integration over all x corresponds to the integration over all lines $(Z_A Z_B)$ in momentum-twistor space.

$$\int d^4x \Longleftrightarrow \int \frac{d^4 Z_A d^4 Z_B}{\text{vol}\,(GL_2) \times \langle \lambda_A \lambda_B \rangle^4} \equiv \int_{AB}$$

The propagators are

$$(x - x_1)^2 \Longleftrightarrow \langle AB\, 12 \rangle \qquad (x - x_2)^2 \Longleftrightarrow \langle AB\, 23 \rangle \qquad etc.$$

and the integral becomes

$$\int_{AB} \frac{\langle 12\, 34 \rangle^2}{\langle AB\, 12 \rangle \langle AB\, 23 \rangle \langle AB\, 34 \rangle \langle AB\, 41 \rangle}$$

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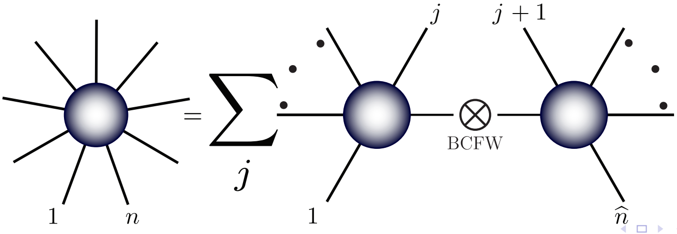

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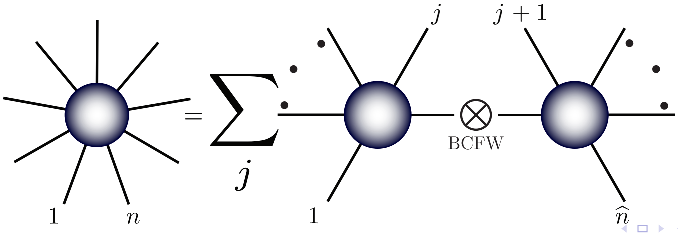

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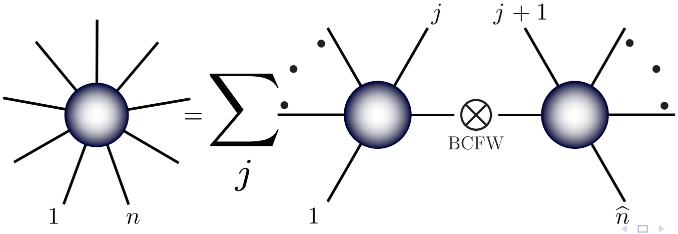

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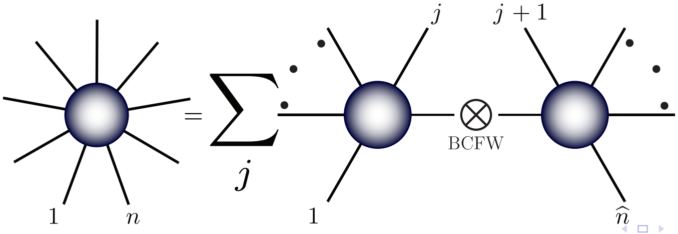

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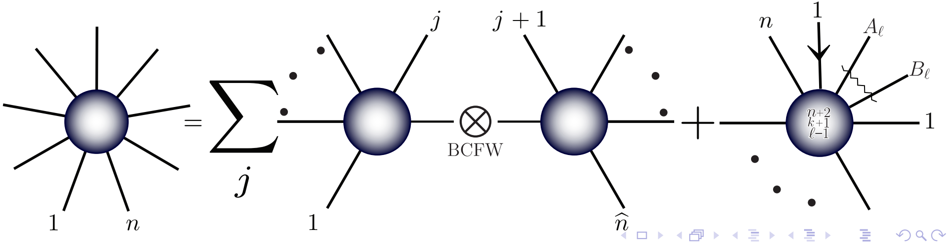

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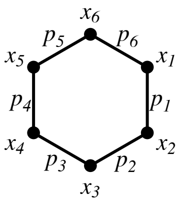

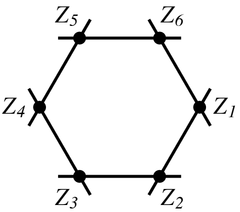

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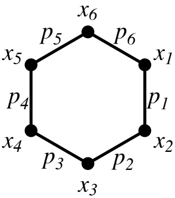

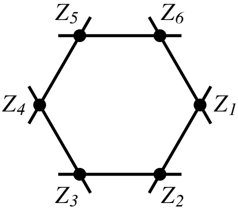

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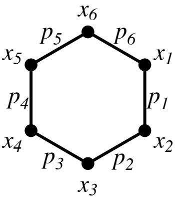

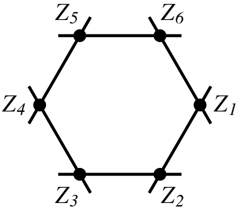

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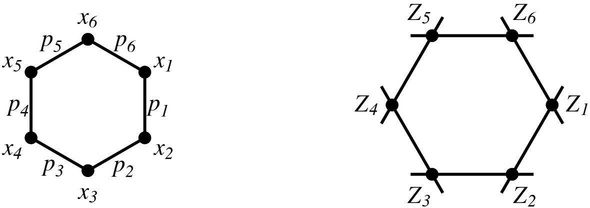

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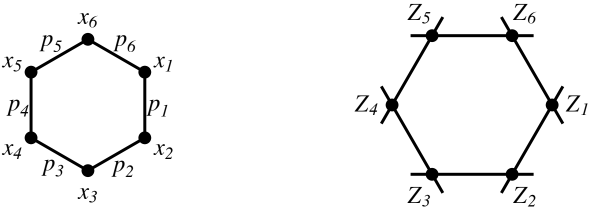

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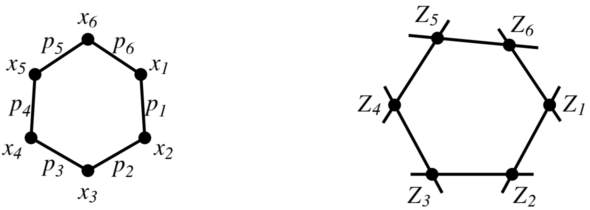

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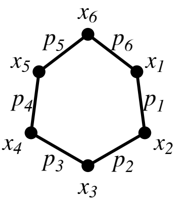

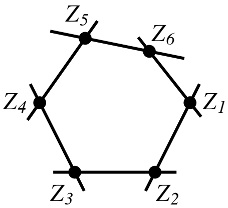

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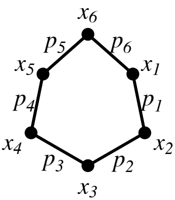

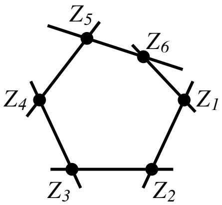

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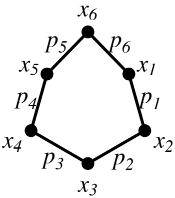

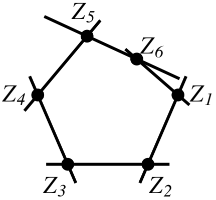

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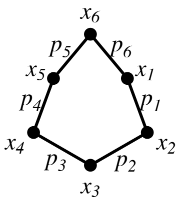

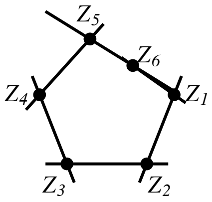

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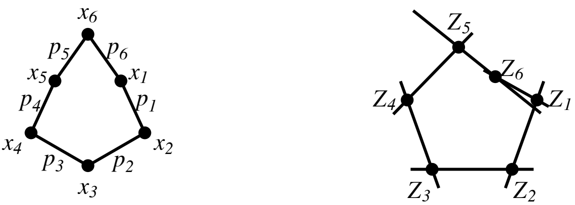

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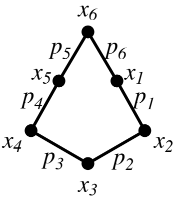

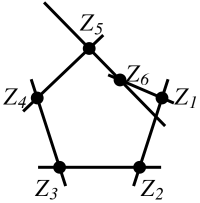

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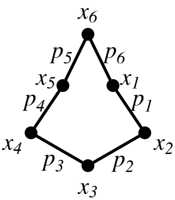

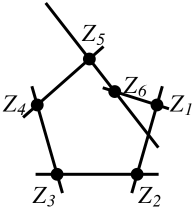

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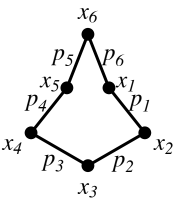

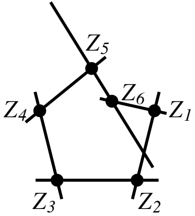

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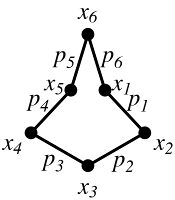

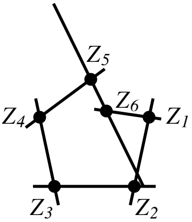

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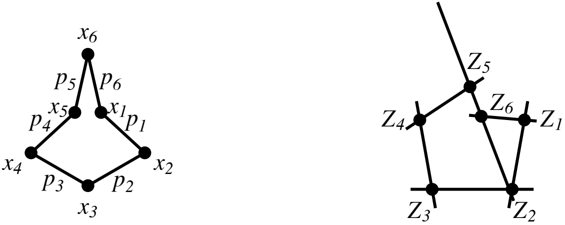

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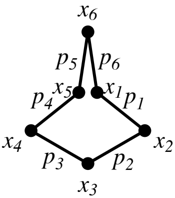

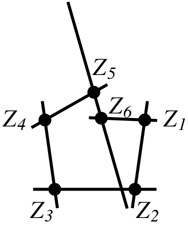

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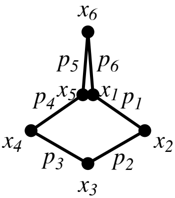

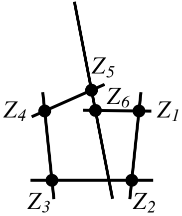

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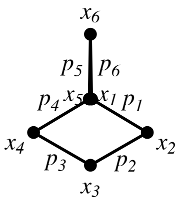

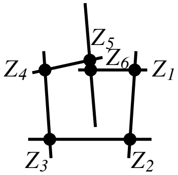

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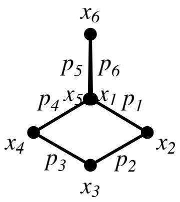

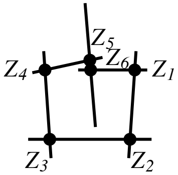

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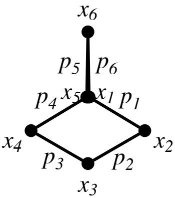

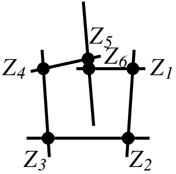

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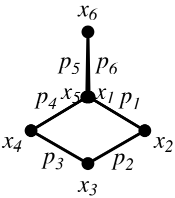

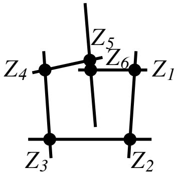

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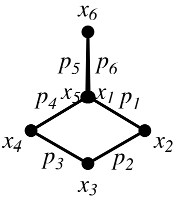

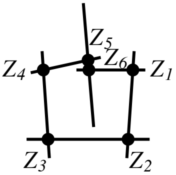

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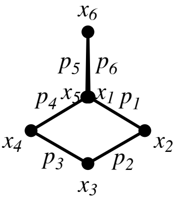

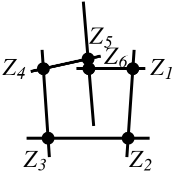

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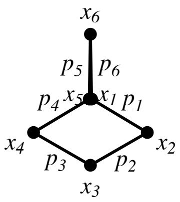
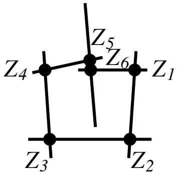
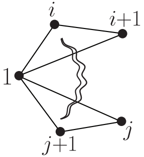




Exempli Gratia: BCFW Form of MHV Loop Amplitudes

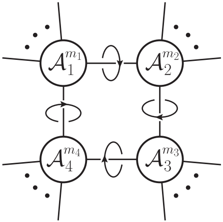
Taking the forward limit of an $(n+2)$-point NMHV tree amplitude we find the following expression for the one-loop MHV amplitude:

$$\sum_{i<j}$$



$$= \sum_{i<j} \int_{AB} \frac{\langle AB\,(1\,i\,i{+}1)\bigcap(1\,j\,j{+}1)\rangle}{\langle AB\,1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,i{+}1\,1\rangle\langle AB\,1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,j{+}1\,1\rangle}$$

Sewing Together Tree Amplitudes in $\mathcal{N} = 4$



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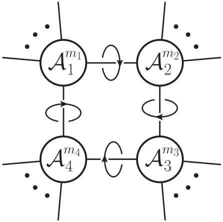
Two-Mass-Easy Schubert Problem



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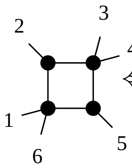
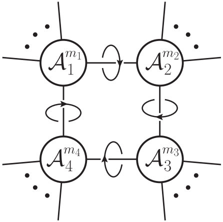


$$\Longleftrightarrow \int\limits_{AB} \frac{\langle 123\,5\rangle\langle 2\,345\rangle}{\langle AB\,12\rangle\langle AB\,23\rangle\langle AB\,45\rangle\langle AB\,56\rangle},$$



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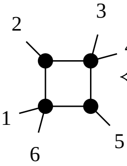

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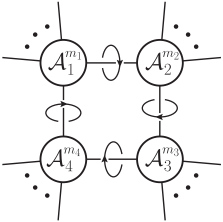

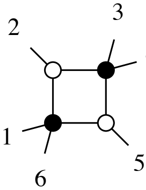

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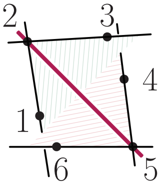
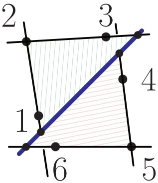


$$(AB) = (25)$$



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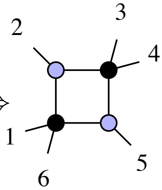
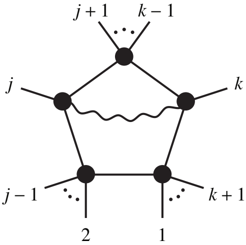


$$(AB) = (123)\bigcap(456)$$

Finite Integrals in Momentum Twistor Space



$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

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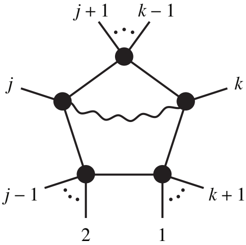

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$$= \quad \mathrm{Li}_2(1-u_1)$$

$$u_1 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j{-}1\,j\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j{-}1\,j\rangle\langle 1\,2\,k{-}1\,k\rangle}$$

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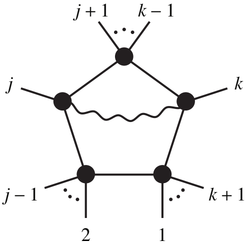
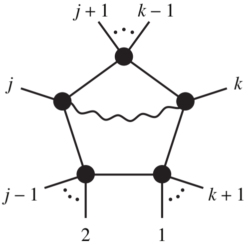


$$\int_{AB} \frac{\langle AB(j{-}1\, j\, j{+}1) \bigcap (k{-}1\, k\, k{+}1)\rangle \langle 1\, 2\, j\, k\rangle}{\langle AB\, 12\rangle \langle AB\, j{-}1\, j\rangle \langle AB\, j\, j{+}1\rangle \langle AB\, k{-}1\, k\rangle \langle AB\, k\, k{+}1\rangle}$$

$$= \quad \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2)$$

$$u_1 \equiv \frac{\langle k\, k{+}1\, 1\, 2\rangle \langle j{-}1\, j\, k{-}1\, k\rangle}{\langle k\, k{+}1\, j{-}1\, j\rangle \langle 1\, 2\, k{-}1\, k\rangle} \qquad u_2 \equiv \frac{\langle j\, j{+}1\, k\, k{+}1\rangle \langle 1\, 2\, j{-}1\, j\rangle}{\langle j\, j{+}1\, 1\, 2\rangle \langle k\, k{+}1\, j{-}1\, j\rangle}$$

Finite Integrals in Momentum Twistor Space



$$\int_{AB} \frac{\langle AB(j{-}1\,j\,j{+}1)\bigcap(k{-}1\,k\,k{+}1)\rangle\langle 1\,2\,j\,k\rangle}{\langle AB\,12\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,k{-}1\,k\rangle\langle AB\,k\,k{+}1\rangle}$$

$$= \quad \mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)-\mathrm{Li}_2(1-u_3)$$

$$u_1 \equiv \frac{\langle k\,k{+}1\,1\,2\rangle\langle j{-}1\,j\,k{-}1\,k\rangle}{\langle k\,k{+}1\,j{-}1\,j\rangle\langle 1\,2\,k{-}1\,k\rangle} \qquad u_2 \equiv \frac{\langle j\,j{+}1\,k\,k{+}1\rangle\langle 1\,2\,j{-}1\,j\rangle}{\langle j\,j{+}1\,1\,2\rangle\langle k\,k{+}1\,j{-}1\,j\rangle}$$

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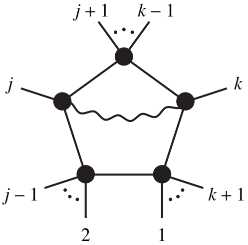

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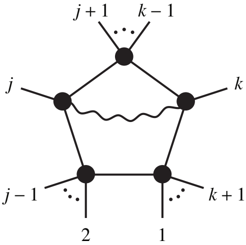

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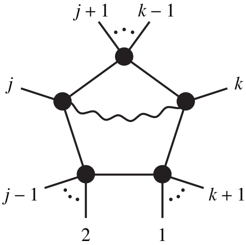

$$\int_{AB} \frac{\langle AB(j{-}1\, j\, j{+}1) \bigcap (k{-}1\, k\, k{+}1)\rangle \langle 1\, 2\, j\, k\rangle}{\langle AB\, 12\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, j\, j{+}1\rangle\langle AB\, k{-}1\, k\rangle\langle AB\, k\, k{+}1\rangle}$$

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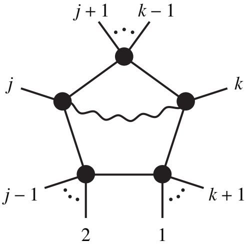

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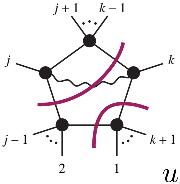

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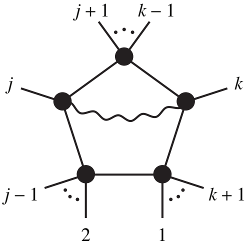

$$\int_{AB} \frac{\langle AB(j{-}1\, j\, j{+}1) \bigcap (k{-}1\, k\, k{+}1)\rangle \langle 1\, 2\, j\, k\rangle}{\langle AB\, 12\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, j\, j{+}1\rangle\langle AB\, k{-}1\, k\rangle\langle AB\, k\, k{+}1\rangle}$$

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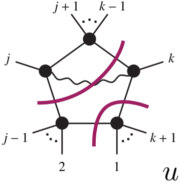

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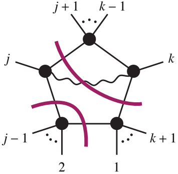

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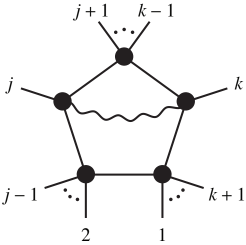

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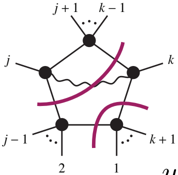

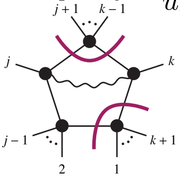

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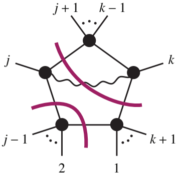

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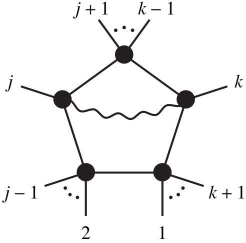

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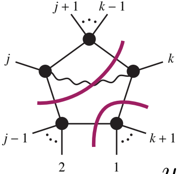

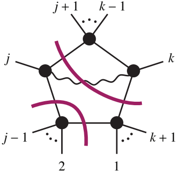

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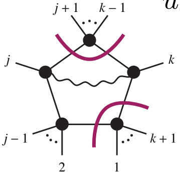

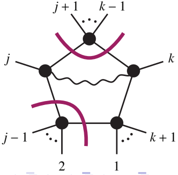

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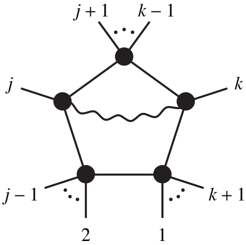

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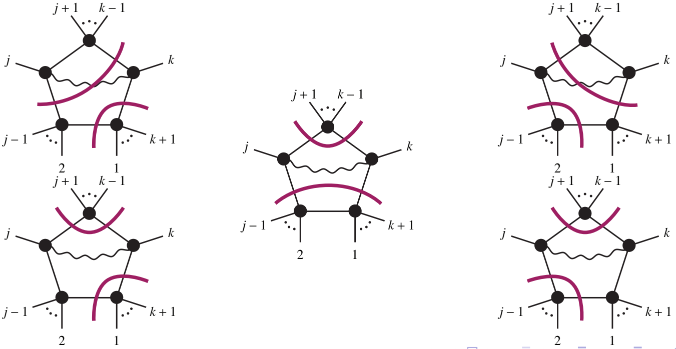

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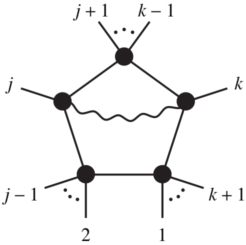

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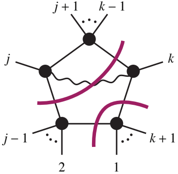

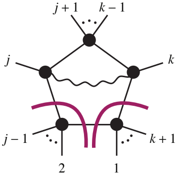

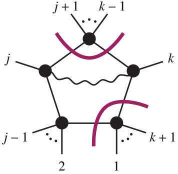

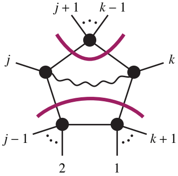

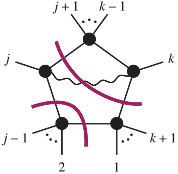

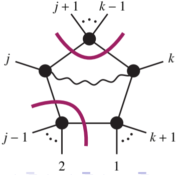

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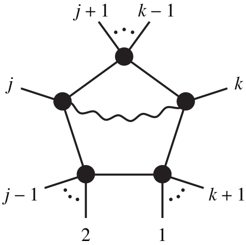

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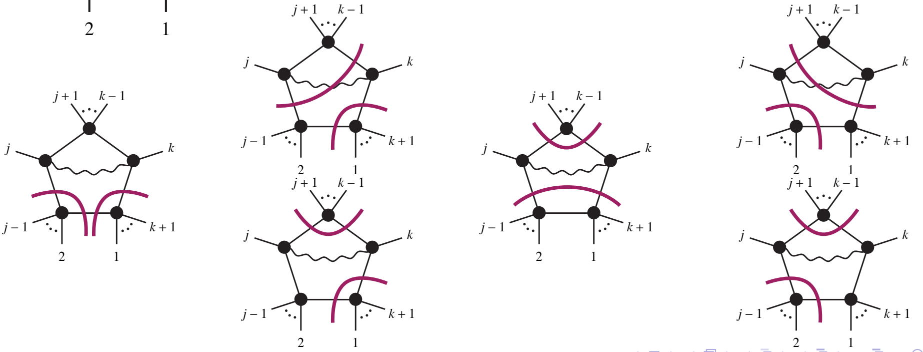

The Continuation of this Logic Through 3-Loops:

In recent months, similar simplifications have been 'guessed' (and checked):

$$\mathcal{A}_n^{(2)}(\ldots, j^-, \ldots, k^-, \ldots) = \frac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

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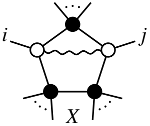

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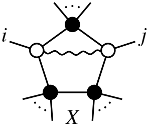
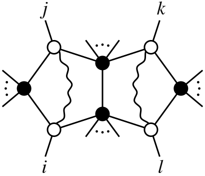


$$+\;\frac{1}{2}\sum_{i<j<k<l<i}$$



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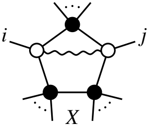
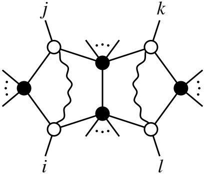
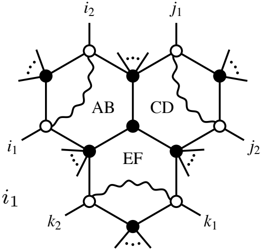
$$\times \Bigg\{\; 1 \;+\; \sum_{i<j<i}$$



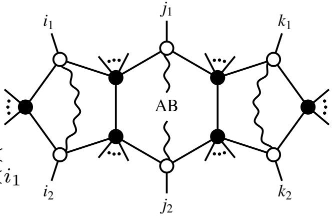
$$+\;\frac{1}{2}\sum_{i<j<k<l<i}$$



$$+\;\frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq\\ \leq j_2<k_1\leq k_2<i_1}}$$



$$+\;\frac{1}{2}\sum_{\substack{i_1\leq j_1<k_1<\\ <k_2\leq j_2<i_2<i_1}}$$



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