Indecomposability in CFT: a pedestrian approach from lattice models

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Joint work with J.L. Jacobsen and H. Saleur
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What is indecomposability?

- Indecomposability $\Leftrightarrow$ Jordan cells

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad\qquad D = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

- A simple example of an indecomposable CFT: symplectic fermions

$$\mathcal{S} = \frac{1}{4\pi} \int d^2 z \, \epsilon_{ab} \, \partial \chi^a \, \bar{\partial} \chi^b$$

gives the OPE $\chi^a \chi^b \sim -\epsilon^{ab} \log |z - w|^2$

- In CFT: indecomposable dilatation operator $\Rightarrow$ logarithmic correlation functions

A little bit of history

The origins

- 1992: study of the Alexander Conway polynomial [Rozansky & Saleur]
- 1992-93: work on dense polymers exhibited logarithmic features at $c = -2$ [Saleur, Kausch]
- 1993: crucial observation by Gurarie: non-trivial theories at $c = 0$ must exhibit logarithmic features.

Developments

- algebraic developments in the late 1990s (for example [Nahm, Gaberdiel & Kausch])
- ... not much progress

Modern perspectives

- progress comes from concrete models: conformal supersigma models and/or scaling limit of concrete lattice models
- and modern indecomposable algebra.

Motivations

- High energy community:
  supersigma models, AdS/CFT, ...?

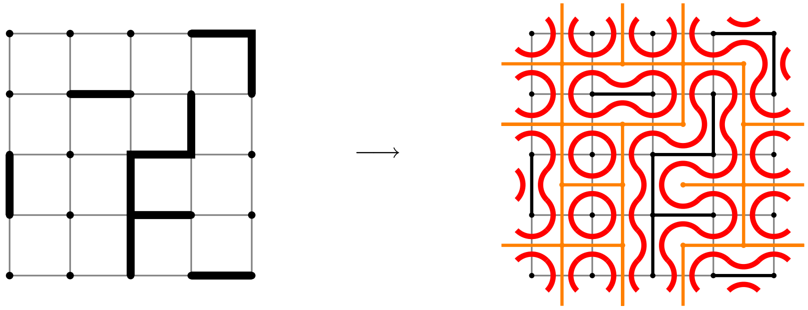
- Condensed matter community:
  - geometrical aspects of 2D phase transitions, polymers, percolation
  - statistical models with quenched disorder (e.g random bond Ising model)
  - spin quantum Hall effect
  - transitions between integer quantum Hall effects
  - ...

Lattice Loop Models

We are particularly interested in $c = 0$ loop models

- dilute polymers (i.e. critical $O(n)$ model in the $n \to 0$ limit)
- critical percolation (i.e. Potts model in the $Q \to 1$ limit)

What is a lattice loop model?

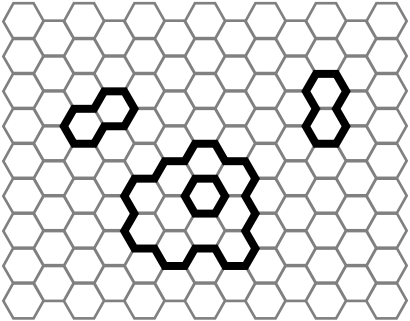


Bond percolation ↔ loops

$$Z = \sum_{\text{conf.}} p^{\#\text{ filled bonds}} (1-p)^{\#\text{ empty bonds}} = 1$$

Critical point corresponds to $p = 1/2$.

What is a lattice loop model?



Boltzmann weight of a configuration

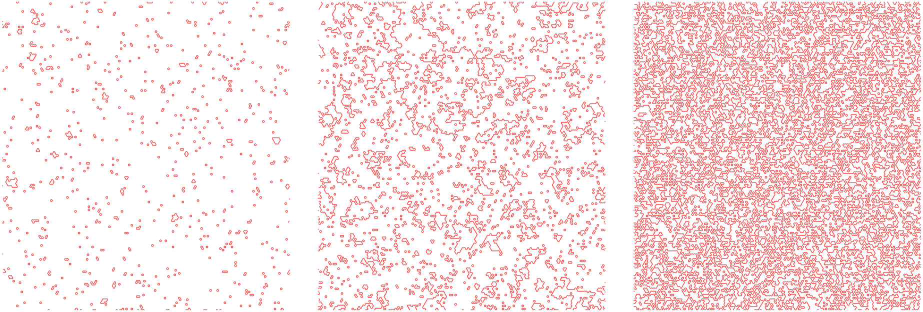
$$x^{\#\text{ links}} \; n^{\#\text{ loops}}$$

Dilute polymers correspond to

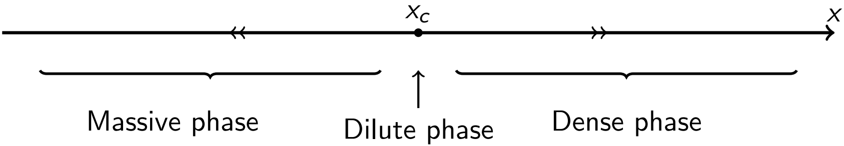
$$n \to 0 \qquad x = x_c \quad \text{(critical coupling)}$$

The O(n) loop model $-2 < n \leq 2$

Massive Dilute Dense

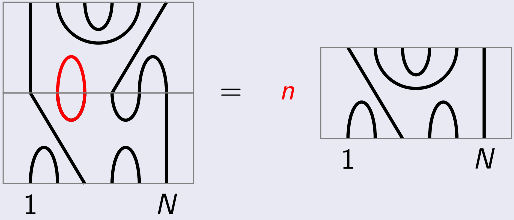


RG flow:

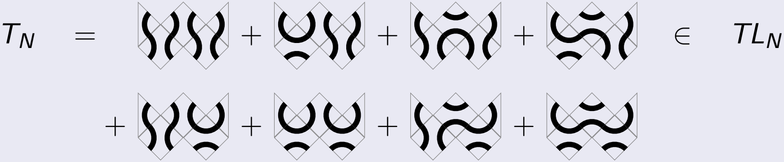


Temperley-Lieb algebra

TL_N : algebra of diagrams with a parameter n



Transfer matrix for the dense loop model

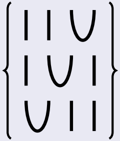


A little bit of representation theory

For generic values of n, the standard modules are irreducible

Standard modules over TL

$\mathcal{V}_0 =$ 

$\mathcal{V}_2 =$ 

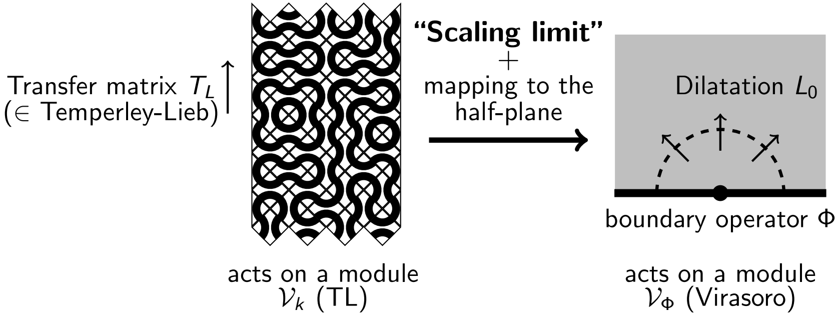
$\mathcal{V}_4 =$ 

We are interested in non-generic cases. For example, for $n = 1$:

so there is a inclusion $\mathcal{V}_0 \leftarrow \mathcal{V}_4$. In general (for $n = 1$)

$$0 \leftarrow \mathcal{V}_k \leftarrow \mathcal{V}_{4-k} \leftarrow \mathcal{V}_{6+k} \leftarrow \mathcal{V}_{10-k} \leftarrow \ldots$$

Irreducible modules are quotients: $\langle k \rangle = \mathcal{V}_k / \mathrm{Im}\,(\mathcal{V}_{4-k})$



Temperley-Lieb $\leftrightarrow$ **Virasoro** correspondance . . .

- In particular, $\mathcal{V}_k \longrightarrow \mathcal{V}_{1,1+k}/\mathcal{V}_{1,-1-k}$ in the scaling limit.
- $\langle k \rangle$: $0 \leftarrow \mathcal{V}_{1,1+k}/\mathcal{V}_{1,-1-k} \leftarrow \mathcal{V}_{1,1+4-k}/\mathcal{V}_{1,-1-4+k} \leftarrow \ldots$

$\langle k \rangle$ is a Feigin-Fuchs (Rocha-Caridi) module!

TL / Virasoro correspondance

Some aspects of the TL/Virasoro correspondance in lattice/continuum loop models:

Spectrum generating algebra	Temperley-Lieb	Virasoro
Parameter	n	c
Non generic cases	$n = q + q^{-1}$ q root of unity	$c = 1 - \frac{6(p-q)^2}{pq}$ p, q integers
Modules	$\mathcal{V}_k$	$\mathcal{V}_\phi$
Adjoint (scalar product)	$e_i^\dagger = e_i$	$L_n^\dagger = L_{-n}$
Determinants	Gram det.	Kac det.
Restrictions	RSOS models	Minimal models

Relation with spin chains

TL algebra in the spin $\frac{1}{2}$-XXZ spin chain

$$\underbrace{\uparrow \otimes \downarrow \otimes \downarrow \otimes \cdots \otimes \uparrow \otimes \downarrow \otimes \uparrow}_{L}$$

- $\mathcal{H}_{XXZ} = (1/2)^{\otimes L}$
- anisotropic Heisenberg coupling

$$H \propto \sum_{i=1}^{L-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right] + \frac{q - q^{-1}}{2} \left[\sigma_1^z - \sigma_L^z \right]$$

- quantum group symmetry $[U_q(SU(2)), H] = 0$

Relation with spin chains

TL algebra in the spin $\frac{1}{2}$-XXZ spin chain

$$\underbrace{\uparrow \otimes \downarrow \otimes \downarrow \otimes \cdots \otimes \uparrow \otimes \downarrow \otimes \uparrow}_{L}$$

- $\mathcal{H}_{XXZ} = (1/2)^{\otimes L}$
- The algebra of projectors over q-singlets is the Temperley-Lieb algebra
- Schur-Weyl duality (here L even)

$$\mathcal{H}_{XXZ} = \bigoplus_{j=0}^{L/2} (2j+1)\, \mathcal{V}_{2j} \qquad [\text{module over TL}]$$

$$= \bigoplus_{j=0}^{L/2} d_j\, (j) \qquad [\text{module over } \mathrm{U_q(SU(2))}]$$

Relation with SUSY models

A simple example: a $gl(1|1)$ chain $(\square \otimes \bar{\square})^{\otimes L}$

$$\underbrace{\square \otimes \bar{\square} \otimes \cdots \otimes \square \otimes \bar{\square}}_{2L}$$

formed by the fundamental rep. $\square$ and its adjoint $\bar{\square}$. Represented by fermionic operators

$$\{f_i, f_j\} = 0 \qquad \left\{f_i, f_j^\dagger\right\} = (-1)^i \delta_{ij}$$

Coupling between nearest neighbours

$$e_i = (f_i^\dagger + f_{i+1}^\dagger)(f_i + f_{i+1})$$

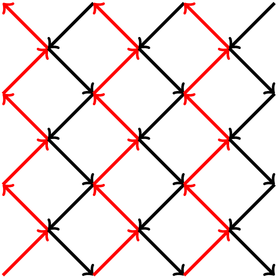
which obey (with $n = 0$)

$$\begin{aligned} e_i^2 &= n\, e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \qquad (|i-j| > 1) \end{aligned}$$

This is again the TL algebra.

Relation with SUSY models

Important observation: alternating $\Box$ and $\bar{\Box}$ $\Rightarrow$ interpretation as edges with a fixed orientation



This is the lattice of the Chalker-Coddington model (plateau transition in the IQHE).

CFT at $c = 0$

Gurarie's argument and b parameter

Gurarie's b

- Conformal invariance requires the OPE

$$\Phi_h(z)\Phi_h(0) \sim \frac{1}{z^{2h}}\left[1+\frac{2h}{c}z^2\,T(0)\,+\,\dots\right]+\text{other primaries}$$

which is ill-defined when $c \to 0$.

- Solution: combine one primary field with descendants of the identity

$$\Phi_h(z)\Phi_h(0) \sim \frac{1}{z^{2h}}\left[1+\frac{2h}{c}z^2\,T(0)\,+\,z^{\tilde{h}}\Phi_{\tilde{h}}+\dots\right]+\dots$$

such that $\tilde{h} \to 2$ when $c \to 0$

Gurarie's b

- $\Phi_{\tilde{h}}$ is not an eigenstate of L_0 in the limit $c \to$, it can cancel the divergence. The OPE becomes

$$\Phi_h(z)\Phi_h(0) \sim \frac{1}{z^{2h}}\left[1 + \frac{h}{b}z^2\left(\log z\, T(0) + t(0)\right) + \ldots\right] + \ldots$$

  where $t(z)$ is a combination of $T(z)$ and $\Phi_{\tilde{h}}(z)$, and $b = -\left(2\frac{\partial \tilde{h}}{\partial c}\right)^{-1}$.
- L_0 is not diagonalizable any more

$$L_0\,|T\rangle = 2\,|T\rangle \qquad L_0\,|t\rangle = 2\,|t\rangle + |T\rangle$$

- b appears in correlation functions

$$\begin{aligned}\langle T(z)T(0)\rangle &= 0\\ \langle T(z)t(0)\rangle &= \frac{b}{z^4}\\ \langle t(z)t(0)\rangle &= \frac{-b\log z + cst}{z^4}\end{aligned}$$

Gurarie's b parameter:

how does it appear in physical quantities?

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Who knows. . . ??

Gurarie's b parameter:

at least, can one compute it in some more concrete models?

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Yes we can.

$\longrightarrow$ [JD, Jacobsen, Saleur 10]

The measure of b in lattice loop models

The measure of b in lattice loop models

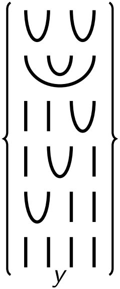
$$b = \langle t | T \rangle$$

- where is the Jordan cell of $|T\rangle$ and $|t\rangle$ in the lattice model?
- how do we normalize $|T\rangle$ and $|t\rangle$?

Jordan cells in the lattice model

TL algebra at $n = 1$

$\mathcal{V} =$



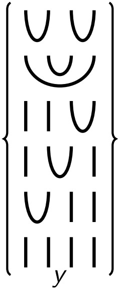
$$e_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Jordan cells in the lattice model

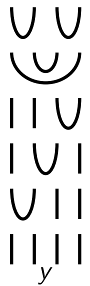
TL algebra at $n = 1$

$\mathcal{V} =$ 

$$P_y = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{y-2}{y-1} & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{y-1} & 0 & 0 & 0 \end{pmatrix}$$

Jordan cells in the lattice model

TL algebra at $n = 1$

$\mathcal{V} = \left\{ \right.$  $\left. \right\}$

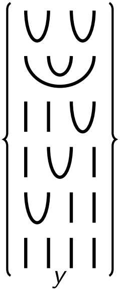
$$P_y^{-1} e_1 P_y = \begin{pmatrix} 0 & 1 & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 1 & 1 \end{pmatrix}$$

$$P_y^{-1} e_2 P_y = \begin{pmatrix} 0 & & & & & \\ & 1 & 1 & & & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & & & 1 & 1 & 1 \\ & & & 0 & 0 & 0 \end{pmatrix}$$

$$P_y^{-1} e_3 P_y = \begin{pmatrix} 0 & 1 & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & 1 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix}$$

Jordan cells in the lattice model

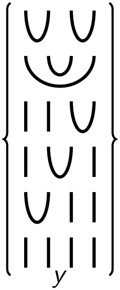
TL algebra at $n = 1$

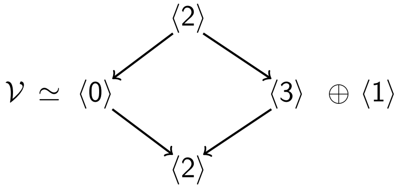
$\mathcal{V} =$ 

$$\mathcal{V} \simeq \begin{array}{ccc} & & \langle 2 \rangle \\ & \swarrow & \\ \langle 0 \rangle & & \\ & \searrow & \\ & & \langle 2 \rangle \end{array} \qquad \oplus \; \langle 1 \rangle$$

Jordan cells in the lattice model

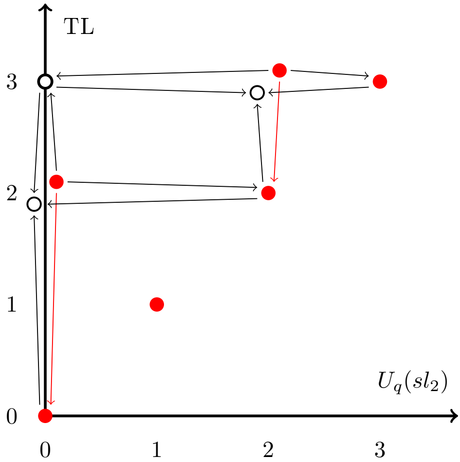
TL algebra at $n = 1$

$\mathcal{V} =$ 



for larger sizes

This is of course what is expected in the XXZ/SUSY rep.



Structure of the XXZ spin chain at $q = e^{i\pi/3}$ on $L = 6$ sites
[Read & Saleur 07]

Jordan cells in the lattice model

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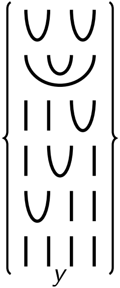
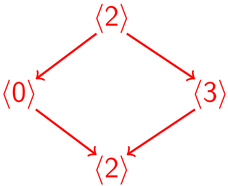


Hamiltonian $H = -\sum_i e_i$
can be put in Jordan form

$$H = \left(\begin{array}{ccc|ccc} E_0 & & & & & \\ & E_1 & 1 & & & \\ & & E_1 & & & \\ \hline & & & E_2 & & \\ & & & & E_3 & \\ & & & & & E_5 \end{array}\right)$$

$\mathcal{V} \simeq$



$\oplus \langle 1 \rangle$

Jordan cells in the lattice model

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$$H = \left(\begin{array}{ccc|ccc} E_0 & & & & & \\ & E_1 & 1 & & & \\ & & E_1 & & & \\ \hline & & & E_2 & & \\ & & & & E_3 & \\ & & & & & E_5 \end{array}\right)$$

Look at $|1\rangle$ and $|\tilde{1}\rangle$ such that

$$\begin{aligned} H\,|1\rangle &= E_1\,|1\rangle \\ H\,|\tilde{1}\rangle &= E_1\,|\tilde{1}\rangle + |1\rangle \end{aligned}$$

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$$\begin{aligned} H\,|1\rangle &= E_1\,|1\rangle \\ H\,|\tilde{1}\rangle &= E_1\,|\tilde{1}\rangle + |1\rangle \end{aligned}$$

...analogous to

$$\begin{aligned} L_0\,|T\rangle &= 2\,|T\rangle \\ L_0\,|t\rangle &= 2\,|t\rangle + |T\rangle \end{aligned}$$

The lattice/continuum identification

$$\begin{aligned} |1\rangle &\leftrightarrow |T\rangle \\ |\tilde{1}\rangle &\leftrightarrow |t\rangle \end{aligned}$$

makes sense because

$$H - E_0 \underset{L\to\infty}{\simeq} \frac{\pi v_F}{L} L_0$$

Great! Let's measure $\langle \tilde{1}|1\rangle$ then...

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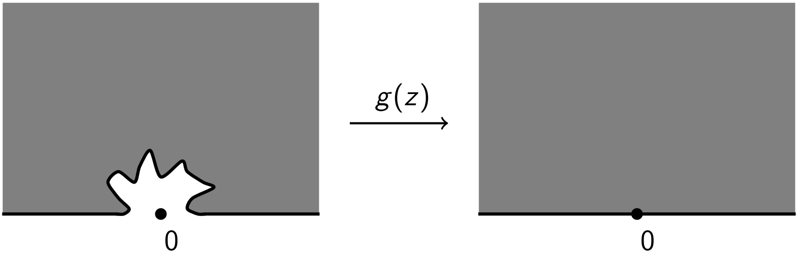
Great! Let's measure $\langle\tilde{1}|1\rangle$ then...

...but the Jordan cell is invariant under global rescaling $|1\rangle \mapsto \alpha\,|1\rangle$ and $|\tilde{1}\rangle \mapsto \alpha\,|\tilde{1}\rangle$. There is no obvious way of fixing the normalization because $\langle 1|1\rangle = \langle T|T\rangle = 0$.

Normalization and the trousers trick

An idea from SLE/CFT work (Cardy, Bauer & Bernard, ...): let's play with the shape of the boundary.
Conformal mapping $z \mapsto g(z)$



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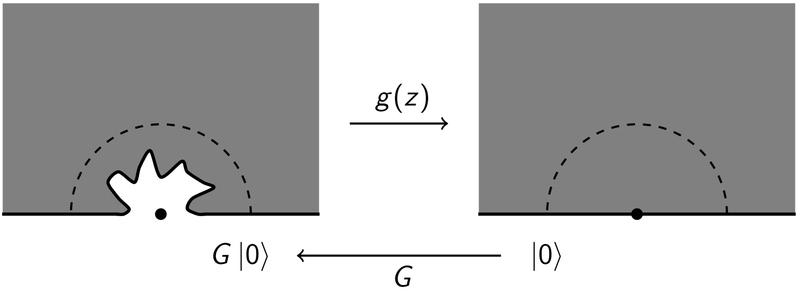

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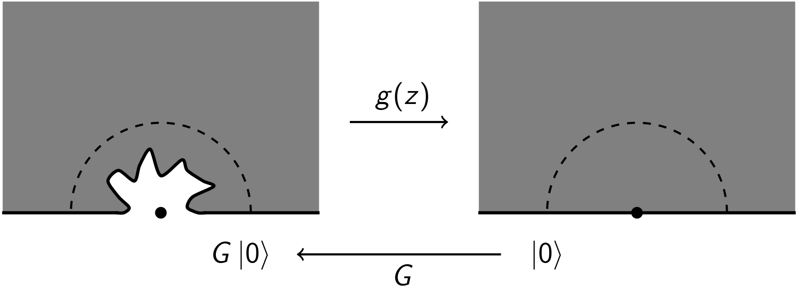
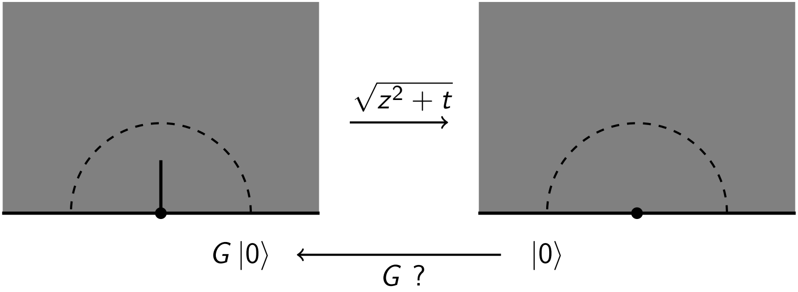


When the mapping g is infinitesimal, $G \in \mathit{Virasoro}^{-}$.

Normalization and the trousers trick

An idea from SLE/CFT work (Cardy, Bauer & Bernard, ...): let's play with the shape of the boundary.

Question: when $g(z) = \sqrt{z^2+1}$, what is G?



$$g_t(z) \;=\; dg \circ dg \circ \cdots \circ dg \circ dg(z)$$

with each dg infinitesimal, corresponding to $dG = -\frac{dt}{2} L_{-2}$.

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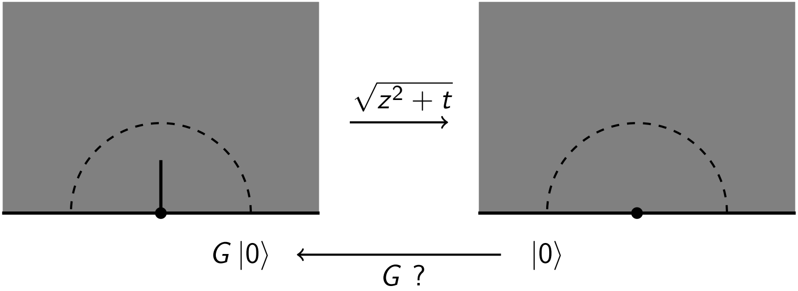
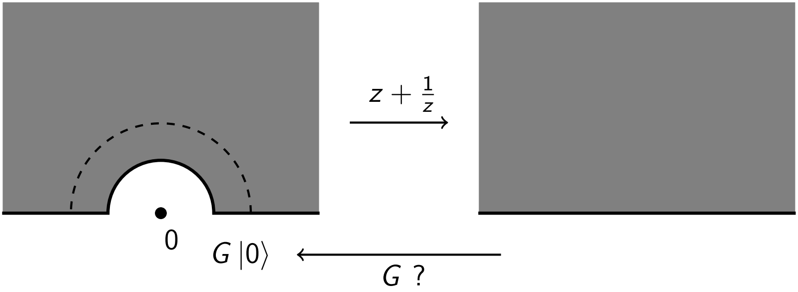


One finds

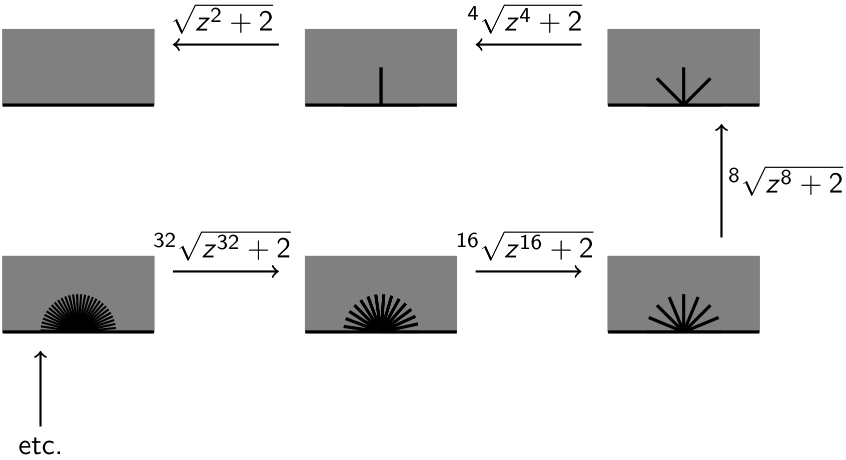
$$G\,|0\rangle \;=\; e^{-\frac{1}{2}L_{-2}}\,|0\rangle \;=\; |0\rangle - \frac{1}{2}L_{-2}\,|0\rangle \;+\; \ldots$$

Normalization and the trousers trick

One can extend the trick to the case $g(z) = z + \frac{1}{z}$

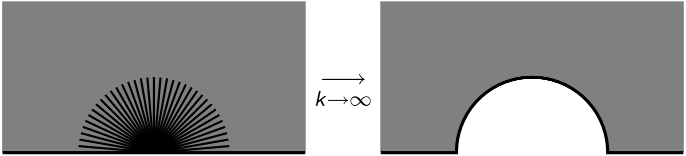


Sequence of conformal mappings



Sequence of conformal mappings

When the number $k = 2^n - 1$ of branches goes to infinity

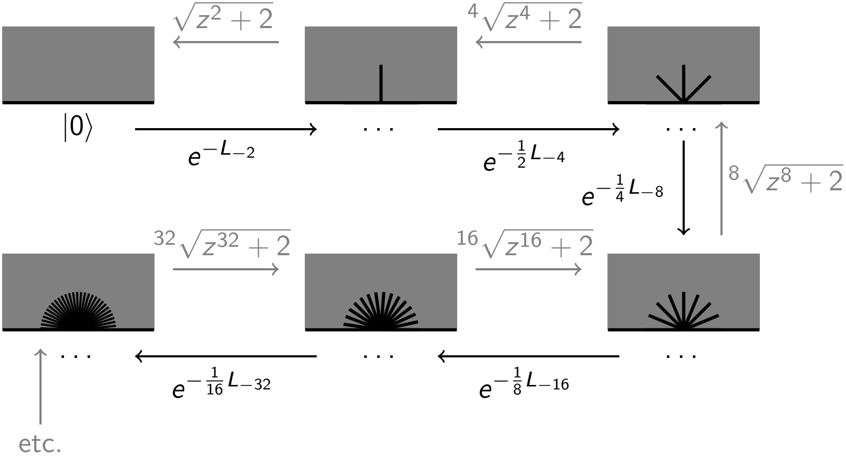


or in other words

$$\lim_{n \to \infty} \sqrt{2 + \sqrt{2 + \ldots \sqrt{2 + z^{2^n}}}} = z + \frac{1}{z}$$

for $|z| > 1$.

Sequence of conformal mappings



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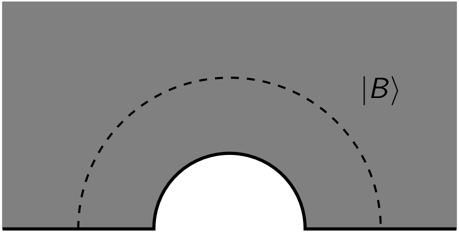
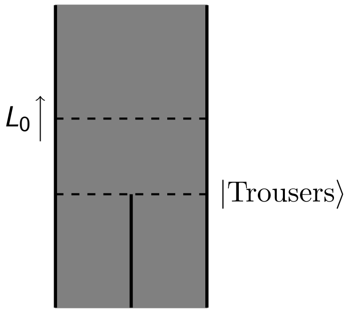
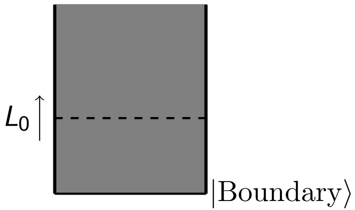


$$|\mathrm{Boundary}\rangle = \lim_{n\to\infty}\left(e^{-\frac{1}{2^{n-1}}L_{-2^n}}\ldots e^{-\frac{1}{2}L_{-4}}e^{-L_{-2}}\right)|0\rangle$$
$$= |0\rangle - L_{-2}|0\rangle + \ldots$$

Normalization and the trousers trick

Mapping those two geometries onto the infinite strip, we get

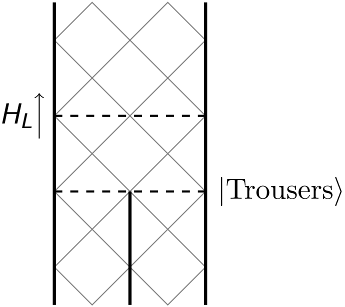


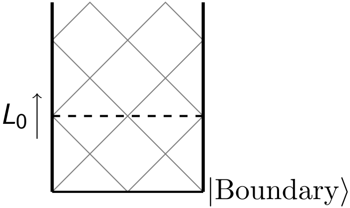


$$|\text{Trousers}\rangle = |0\rangle - \frac{1}{2}\,|T\rangle + \ldots \qquad |\text{Boundary}\rangle = |0\rangle - |T\rangle + \ldots$$

Normalization and the trousers trick

The states $|\mathrm{Trousers}\rangle$ and $|\mathrm{Boundary}\rangle$ can be built on the lattice





$$|\mathrm{Trousers}\rangle = |0\rangle_{L/2} \otimes |0\rangle_{L/2} \qquad |\mathrm{Boundary}\rangle = |\cup\ \cup \ldots \cup\ \cup\rangle$$

Normalization and the trousers trick

Now one can build the quantities

$$b_{\text{Trous.}} = 4\frac{\langle \text{Trous.}|\tilde{1}\rangle \langle \tilde{1}|\text{Trous.}\rangle}{\langle 1|\tilde{1}\rangle} \quad b_{\text{Bound.}} = \frac{\langle \text{Bound.}|\tilde{1}\rangle \langle \tilde{1}|\text{Bound.}\rangle}{\langle 1|\tilde{1}\rangle}$$

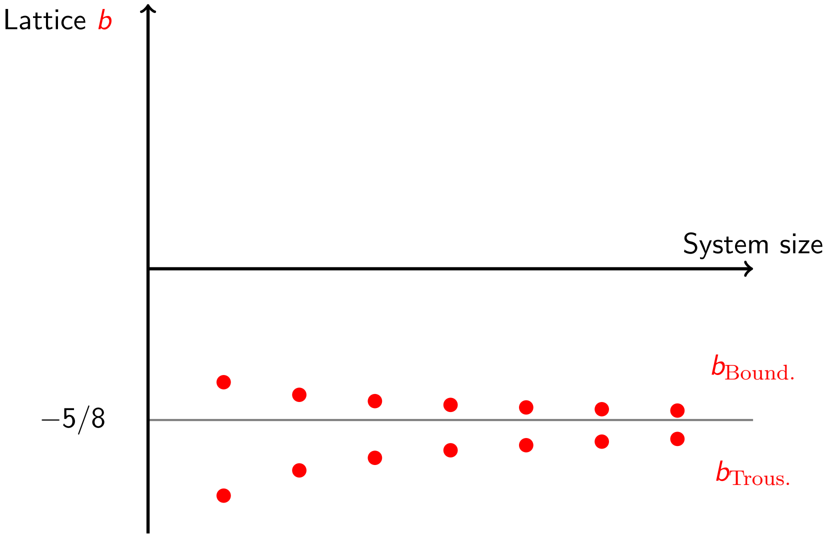
which are both invariant under global rescaling

$$\begin{aligned} |1\rangle &\mapsto \alpha\,|1\rangle \\ |\tilde{1}\rangle &\mapsto \alpha\,|\tilde{1}\rangle \end{aligned}$$

and are both expected to converge to b in the thermodynamic limit.

Results

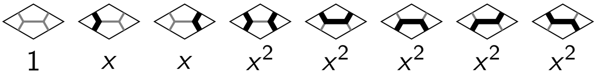
Schematic plot of the results



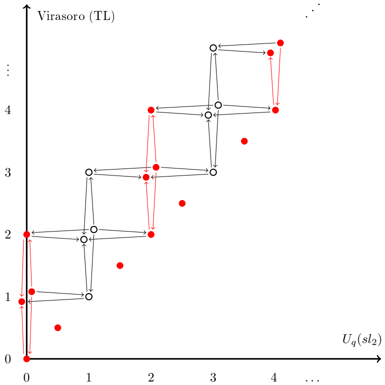
Dilute polymers

Basically, the same story.

- Transfer matrix ($x = x_c$)



- Temperley-Lieb algebra at $n = 0$



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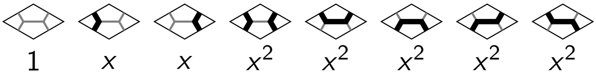
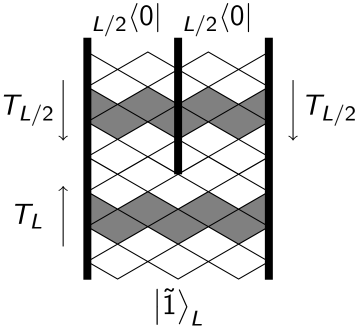


- Temperley-Lieb algebra at $n = 0$
- Trousers/Boundary tricks



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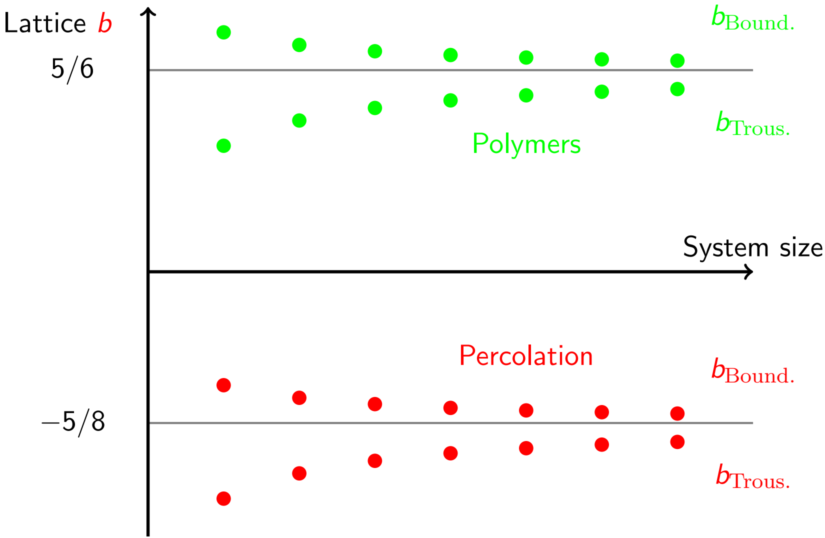
Schematic plot of the results



Conclusion

- First lattice realization and measure of indecomposability b parameters [JD, JJ, HS 10].
- Somehow, not satisfying because relies on particular tricks $\rightarrow$ generalization not obvious
- Generalization and systematic study to appear soon [R. Vasseur, J. Jacobsen, H. Saleur 11]
- What about the periodic case? How does it mix Vir and $\bar{Vir}$?

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…and what is the physics hidden behind these parameters?

Which kind of (interesting) observables are they related to?

Thank you.