One Loop Tests of Higher Spin AdS/CFT

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Massless higher spins

- Consistent interactions of *massless* higher spin fields (gauge fields!) are highly constrained
- In flat space, no consistent theory of interacting massless higher spin fields of spin $s > 2$ (Coleman-Mandula, Weinberg…)
- However, with non-zero cosmological constant, Vasiliev explicitly constructed consistent fully non-linear theories of interacting massless higher spin fields (in arbitrary dimensions). They involve infinite towers of higher spin fields, including in particular the graviton ($s=2$).

Higher spins and AdS/CFT

- Vasiliev wrote down a set of consistent gauge invariant equations of motion. They admit a vacuum solution which is AdS space (or dS if the cosmological constant is positive. Will focus on AdS case in this talk).
- In the simplest bosonic 4d theory, the linearized spectrum around the AdS vacuum consist of an infinite tower of higher spin fields plus a scalar

$$\begin{array}{lll} \text{Spectrum :} & s = 1, 2, 3, \ldots, \infty & \text{gauge fields} \\ & s = 0, \quad m^2 = -2/\ell_{AdS}^2 & \text{scalar} \end{array}$$

Vasiliev equations

- The Vasiliev equations in 4d

$$d_x \hat{A} + \hat{A} * \hat{A} = f_*(B * K) dz^2 + \overline{f}_*(B * \overline{K}) d\bar{z}^2,$$
$$d_x B + \hat{A} * B - B * \pi(\hat{A}) = 0.$$

- Essentially, *A* contains the metric and all other higher spin fields, and *B* contains the scalar field and the curvatures (weyl tensors) of the HS fields.
- The linearized equations are the standard equations for a scalar, linearized graviton and free massless HS fields (Fronsdal).

Higher spins and AdS/CFT

- From the point of view of AdS/CFT, it is not too surprising that such theories exist.
- Consider a free theory of N free complex scalar fields in 3d

$$S = \frac{1}{2}\int d^3x \partial_\mu \phi_i^* \partial^\mu \phi^i \,, \qquad i = 1, \ldots, N$$

- It has a U(N) global symmetry under which the scalar transforms as a *vector*. (This is different from familiar examples of AdS/CFT, where the CFT side is usually a gauge theory with matrix (adjoint) type fields).

Higher spins and AdS/CFT

- By virtue of being free, it is easy to see that this theory has an infinite tower of conserved HS currents

$$J_{\mu_1 \cdots \mu_s} = \phi_i^* \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^i + \ldots$$
$$\partial^\mu J_{\mu \mu_2 \cdots \mu_s} = 0 \,, \qquad \Delta(J_s) = s + 1$$

- If we consider U(N) invariant operators (*singlet sector*), these currents, together with the scalar operator

$$J_0 = \phi_i^* \phi^i \,, \qquad \Delta = 1$$

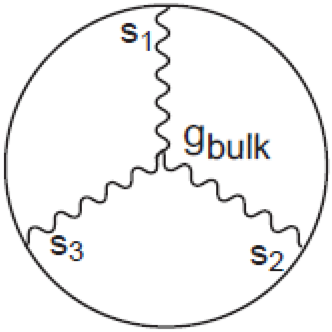
are all the "single trace" primaries of the CFT.

Higher spins and AdS/CFT

- By the usual AdS/CFT dictionary, single trace primaries are dual to single particle states in AdS.
- *Conserved* currents are dual to *gauge* fields (massless HS fields)
- A scalar operator is dual to a bulk scalar with $\Delta(\Delta - d) = m^2$
- This precisely matches the spectrum of Vasiliev's bosonic theory in AdS_4.

Higher spins and AdS/CFT

- The dual higher spin fields are necessarily interacting in order to reproduce the non-vanishing correlation functions of HS currents in the CFT



- The coupling constant scales as $g_{\rm bulk} \sim \dfrac{1}{\sqrt{N}}$ (or in terms of Newton's constant $G_N \sim 1/N$).

- The large N limit corresponds as usual to weak interactions in the bulk.

Higher spins and AdS/CFT

- We could also consider a theory of N free real scalars, and look at the O(N) singlet sector
- The spectrum of single trace primaries now includes a scalar plus all the *even* spin HS currents
- On Vasiliev's theory side, this corresponds to a consistent truncation of the equations which retains only the even spin gauge fields ("Minimal bosonic HS theory").

$$\text{Spectrum}: \qquad s = 2, 4, 6, \ldots, \infty \qquad \text{gauge fields}$$
$$s = 0, \quad m^2 = -2/\ell^2_{AdS} \quad \text{scalar}$$

Higher spins and AdS/CFT

- The conjecture that the (singlet sector of) the O(N)/U(N) vector model is dual to the bosonic Vasiliev theory was made by Klebanov and Polyakov (2002). A crucial observation is that one can also consider the Wilson-Fisher fixed point reached by a relevant "double trace" deformation $\lambda(\phi^i\phi^i)^2$ of the free theory.

- The IR fixed point ("critical vector model") is an *interacting* CFT whose single trace spectrum has a scalar operator of dimension $\Delta = 2 + O(1/N)$ and approximately conserved HS current of dimension

$$\Delta = s + 1 + O(1/N)$$

Higher spins and AdS/CFT

- One can also consider a fermionic version of the vector model

$$S = \frac{1}{2}\int d^3x \bar{\psi}_i \gamma^\mu \partial_\mu \psi^i$$

- The single trace spectrum is similar to the scalar case, there is an infinite tower of conserved HS current, plus a parity odd scalar $\bar{\psi}_i \psi^i$ of dimension 2.
- It turns out that there is indeed a Vasiliev's theory with such a single particle spectrum: at linearized level, the only difference is that the bulk scalar is now a pseudo-scalar. This is called "type B", while the theory dual to scalars "type A".
- At non-linear level "type A" and "type B" theories have different interactions (for instance, the graviton cubic coupling is different).

Higher spins and AdS/CFT

- One can also consider an interacting (UV) fixed point corresponding to a $(\bar{\psi}_i \psi^i)^2$ deformation of the free theory. It corresponds again to alternate boundary conditions in the bulk.
- Summary of higher spin/vector model dualities

	HS A-type	HS B-type
Δ=1 scalar b.c.	Free U(N)/O(N) scalar	Critical U(N)/O(N) fermion
Δ=2 scalar b.c.	Critical U(N)/O(N) scalar	Free U(N)/O(N) fermion

Some comments

- Pure HS gauge theories have exactly the right spectrum to be dual to *vector models* (adjoint theories have many more single trace operators)
- The restriction to singlet sector can be implemented by *gauging* the U(N)/O(N) symmetry, and taking the limit of zero gauge coupling. In 3d, we can do this with Chern-Simons gauge theory.
- In the large N limit with λ=N/k fixed, the singlet sector of the free (critical) theories correspond to the limit λ–>0.

HS/CS vector model dualities

- This point of view has suggested a generalization of the higher spin AdS/CFT duality to the vector models coupled to Chern-Simons theory (SG et al, Aharony, Gur-Ari, Yacoby)
- They were conjectured to be dual to parity breaking versions of Vasiliev's theory in AdS_4.

CS+vector model ←→ parity breaking HS theory

- These theories involve extra parameters which can be mapped to λ and allow an interpolation between "type A" and "type B" theories.
- On the CFT side, this has suggested a novel bose-fermi duality relating theories of bosons coupled to CS to theories of fermions coupled to CS, with N <–> k (generalizes level-rank duality).

Free energy on S^3

- These HS/vector model dualities have been explicitly tested so far at the level of 3-point correlation functions (SG-Yin, Maldacena-Zhiboedov, Didenko-Skorvstov…)
- We would like to make a new type of test based on a different, global, observable of the CFT: the free energy on a round sphere S^3, $F=-\log Z$.
- It is an interesting quantity that for any RG flow satisfies $F_{UV} > F_{IR}$.
- For a CFT, it is also related to the entanglement entropy across a circle.

Free energy on S^3

- In the CFT, it is simply defined as the log of the partition function of the theory on S^3 (generalization to S^d is straightforward)

$$F = -\log Z \qquad\qquad Z = \int D\phi e^{-S}$$

$$S = \int d^3x \sqrt{g} \left(\partial_\mu \phi^i \partial^\mu \phi^i + \frac{R}{8} \phi^i \phi^i \right)$$

- This is straightforward to compute in the free theory: need to evaluate the determinant of the kinetic operator

$$F = \frac{1}{2} \log \det \left(-\nabla^2 + 3/4 \right)$$

Free energy on S^3

- The explicit computation gives (Klebanov, Pufu, Safdi)

$$F = \frac{N}{2}\sum_{n=0}^{\infty}(n+1)^2 \log[(n+1/2)(n+3/2)] = N\left(\frac{\log 2}{8} - \frac{3\zeta_3}{16\pi^2}\right)$$

for N real scalars, and twice this result for N complex scalars.

- One can also perform the calculation in the critical theory (in a large N expansion), with the result

$$F^{\text{critical}} = F^{\text{free}} - \frac{\zeta_3}{8\pi^2} + O(1/N)$$

Free energy on S^3 from the bulk

- The challenge is: can we reproduce these results from the bulk? In particular, can we see the vanishing of the subleading corrections in the large N expansion of the free energy from the HS dual to the free theory?
- How do we compute F from the bulk?

$$Z_{\rm CFT} = Z_{\rm bulk}$$

- We "simply" have to compute the partition function of the Vasiliev's theory on the Euclidean AdS_4 vacuum

$$ds^2 = d\rho^2 + \sinh^2\rho \, d\Omega_3$$

Free energy on S^3 from the bulk

- In practice, we should compute the path integral of the bulk theory, where we expand the metric around AdS_4 and integrate over all quantum fluctuations

$$
\begin{aligned}
Z_{\rm bulk} &= \int D\varphi_{(0)} D g_{\mu\nu} D\varphi_{(s)} e^{-S[g=g_0+h,\varphi_{(0)},\varphi_{(s)}]} \\
&= e^{-\frac{1}{G_N}F^{(0)} - F^{(1)} - G_N F^{(2)} + \ldots} = e^{-F_{\rm bulk}}
\end{aligned}
$$

- Here G_N is Newton's constant, which scales as $1/G_N \sim N$.

Free energy on S^3 from the bulk

- The full explicit bulk action is not known, but is expected to take a form

$$S \sim \frac{1}{G_N} \int d^4x \sqrt{g} \left(R + \Lambda + R^3 + R^4 + \ldots \right.$$
$$\left. + \varphi_{(s)} \Delta_s \varphi_{(s)} + \sum C_{s_1 s_2 s_3} \partial^{k_1} \varphi_{s_1} \partial^{k_2} \varphi_{s_2} \partial^{k_3} \varphi_{s_3} + \ldots \right)$$

- The leading term $\frac{1}{G_N} F^{(0)}$ in the bulk free energy corresponds to evaluating this action on the AdS_4 background metric, with all other fields set to zero.
- This is already very non-trivial, as it requires to know the form of all the higher derivative corrections in the metric sector (we know they are non-trivial from knowledge of correlation functions).

Free energy on S^3 from the bulk

- One would like to show that

$$S_{\text{classical}}[g_{\mu\nu} = AdS_4, \varphi_{(s)} = 0] = \frac{1}{G_N} F^{(0)} = N\left(\frac{\log 2}{4} - \frac{3\zeta_3}{8\pi^2}\right)$$

- This is one of the outstanding open problems in testing HS/vector model dualities.
- While we cannot show this (yet), we can start by something simpler, namely assume that this tree level piece works, and compute the one-loop contribution $F^{(1)}$ to the bulk free energy.

Comments on classical piece

- Before moving to the one-loop calculation, a couple of comments on the calculation of the classical piece.
- The metric part of the action is expected to contain an infinite number of higher derivative corrections. When evaluated on AdS_4 background, they would all reduce to some power of the Ricci scalar, with some coefficients.
- So the on-shell classical action is expected to have a form

$$S_{\text{classical}} = \int d^4x \sqrt{g} \sum_{n=1}^{\infty} c_n R^n$$

- The Ricci scalar R=-12, and if $c_n \sim 1/n + 1/n^3$, this could reproduce the expected numbers log(2) and ζ_3.

Comments on classical piece

- In practice this approach is difficult lacking an explicit action.
- An alternative approach could be to use the fact that F is related to the circle entanglement entropy. In the bulk, we would need a generalization of Ryu-Takayanagi formula to Vasiliev's theory. It is a higher derivative theory, so we would not just compute the area of a minimal surface.
- Perhaps the structure of Vasiliev's equations would suggest a natural gauge invariant 2-form that can be integrated over the 2-dimensional minimal surface.

The one-loop piece

- Let us now concentrate on the calculation of the one-loop contribution $F^{(1)}$ to the bulk free energy

$$e^{-\frac{1}{G_N}F^{(0)}-F^{(1)}-G_N F^{(2)}+\dots} = e^{-F_{\text{bulk}}}$$

- Even if we don't know the full action, we know that the linearized equations correspond to standard kinetic terms for all the higher spin fields, so we assume a canonical quadratic action

$$S_{(2)} = \int d^4x\sqrt{g}\left(\varphi_{(0)}(-\nabla^2-2)\varphi_{(0)} + \sum_{s=1,2,\dots}\varphi_{(s)}\Delta_s\varphi_{(s)}\right)$$

The one-loop piece

- The one-loop free energy is then obtained by computing the log of the determinants for the corresponding operators and summing over all spins
- The HS fields have a linearized gauge invariance $\delta\varphi_{(s)} = D\epsilon_{(s-1)}$ that must be gauge fixed.
- The gauge fixing for spin 1 and 2 is well known. For higher spins, it has been worked out in detail by various authors (Gaberdiel, Grumiller, Saha, Gupta, Lal…)

The one-loop piece

- One can introduce spin s-1 ghosts, then after decomposing physical and ghost fields into their irreducible parts, the contribution to the one-loop free energy of each HS field is the ratio of determinants on symmetric traceless transverse fields

$$\frac{\left[\det_{s-1}^{STT}\left(-\nabla^2+s^2-1\right)\right)\right]^{\frac{1}{2}}}{\left[\det_{s}^{STT}\left(-\nabla^2+s(s-2)-2\right)\right]^{\frac{1}{2}}}$$

- Note that degrees of freedom work. And the “mass terms” correspond to dual CFT dimensions Δ=s+1 for the physical part, and Δ=s+2 for the ghost part (these are the dimensions of the spin s current and its divergence).

One-loop free energy

- We have to compute

$$F_{1-\text{loop}} = \frac{1}{2}\log\det\left(-\nabla^2 - 2\right) + \frac{1}{2}\sum_{s=1}^{\infty}\left[\log\det{}_s\left(-\nabla^2 - 2 + s(s-2)\right) - \log\det{}_{s-1}\left(-\nabla^2 + s^2 - 1\right)\right]$$

- Luckily, a big part of the calculation was already done in a series of papers by Camporesi and Higuchi in the '90's.
- By heat kernel techniques, they computed the determinants of operators of the form $-\nabla^2 + \kappa^2$ acting on STT fields of arbitrary spin.

Spectral zeta function

- More precisely, they computed the *spectral zeta function*. This is related to the heat kernel by a Mellin transform.
- For operators with discrete eigenvalues, the spectral zeta function is defined as

$$\zeta(z) = \sum_n d_n \lambda_n^{-z}$$

- In non-compact spaces such as AdS, this becomes

$$\zeta(z) = \int du \mu(u) \lambda_u^{-z}$$

where $\mu(u)$ is a “spectral density”.

AdS Spectral zeta function

- The explicit spectral zeta function in AdS is

$$\zeta_{(\Delta,s)}(z) = \left(\frac{\int \mathrm{vol}_{AdS_{d+1}}}{\int \mathrm{vol}_{S^d}}\right) \frac{2^{d-1}}{\pi} g(s) \int_0^\infty du \, \frac{\mu_s(u)}{\left[u^2 + \left(\Delta - \frac{d}{2}\right)^2\right]^z}$$

with $\Delta(\Delta - d) - s = \kappa^2$

- In the present case of d=3

$$\mathrm{vol}_{AdS_4} = \frac{4}{3}\pi^2 \,, \qquad \mathrm{vol}_{S^3} = 2\pi^2$$

$$\mu_s(u) = \frac{\pi u}{16}\left[u^2 + \left(s + \frac{1}{2}\right)^2\right] \tanh \pi u \,, \qquad g(s) = 2s + 1$$

AdS Spectral zeta function

- In terms of the spectral zeta function, the contribution to the one-loop free energy is then obtained as

$$F^{(1)}_{(\Delta,s)} = -\frac{1}{2}\zeta'_{(\Delta,s)}(0) - \frac{1}{2}\zeta_{(\Delta,s)}(0)\log\left(\ell^2\Lambda^2\right)$$

- Importantly, in every even dimensional bulk spacetime, there is a logarithmic divergence proportional to the value of the spectral zeta function at z=0. It is related to the conformal anomaly.

UV finiteness

- For the duality to be exact, this divergence should not be present in the full HS theory: the bulk theory should be finite.
- While each spin contributes a log divergence, can the divergence cancel in the sum over the infinite tower of fields?

$$
\begin{aligned}
F^{(1)}\Big|_{\log-\mathrm{div}} &= -\frac{1}{2}\left(\zeta_{(1,0)}(0)+\sum_{s=1}^{\infty}\left(\zeta_{(s+1,s)}(0)-\zeta_{(s+2,s-1)}(0)\right)\right)\log\left(\ell^2\Lambda^2\right) \\
&= \left(\frac{1}{360}+\sum_{s=1}^{\infty}\left(\frac{1}{180}-\frac{s^2}{24}+\frac{5s^4}{24}\right)\right)\log\left(\ell^2\Lambda^2\right)
\end{aligned}
$$

UV finiteness

- It appears natural to regulate this sum with the usual Riemann zeta-function regularization (we will come back to the question of regularization later). Recall that ζ(0)=-1/2, and ζ(-2)=ζ(-4)=0. So

$$\frac{1}{360}+\sum_{s=1}^{\infty}\left(\frac{1}{180}-\frac{s^2}{24}+\frac{5s^4}{24}\right)=0$$

- So Vasiliev's theory is one-loop finite! (at least accepting this regularization).
- The same result holds for the theory with even spins only, and regardless of boundary conditions on the scalar.
- This is a simple but important consistency test.

The finite part

- Having shown that the log divergence cancels, we can move on to the computation of the finite contribution. This is considerably more involved. Computing the derivative of the spectral zeta-function, the result is expressed as

$$F^{(1)} = -\frac{1}{2}\mathcal{I}(-1/2, 0) - \frac{1}{2}\sum_{s=1}^{\infty}\left[\mathcal{I}(s-1/2, s) - \mathcal{I}(s+1/2, s-1)\right]$$

with:

$$\mathcal{I}(\nu, s) = \frac{1}{3}(2s+1)\int_0^{\nu} dx \left[\left(s+\frac{1}{2}\right)^2 x - x^3\right]\psi(x+\frac{1}{2})$$

The finite part

- After a somewhat lengthy calculation we find

$$\mathcal{I}\left(-\frac{1}{2},0\right) = -\frac{1}{3}\int_{-1/2}^{0} dx\left(\frac{x}{4}-x^3\right)\psi(x+\frac{1}{2}) = \frac{11}{1152} - \frac{11\log 2}{2880} - \frac{\log A}{8} - \frac{\zeta(3)}{8\pi^2} + \frac{5\zeta'(-3)}{8}$$

$$\sum_{s=1}^{\infty}\left[\mathcal{I}\left(s-\frac{1}{2},s\right) - \mathcal{I}\left(s+\frac{1}{2},s-1\right)\right]$$
$$= -\frac{11}{1152} + \frac{11\log 2}{2880} + \frac{\log A}{8} - \frac{5\zeta'(-3)}{8} - \frac{\zeta'(-2)}{2}$$

- Recalling that $\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}$, the higher spin tower precisely cancels the scalar!

The finite part

- So we conclude that the one-loop bulk free energy in Vasiliev's type A theory with $\Delta=1$ boundary condition for the scalar is exactly zero

$$F^{(1)} = 0$$

- This is precisely consistent with the fact that in the dual free CFT the large N expansion should be trivial.

Δ=2 and the critical vector model

- We can also easily do the calculation with Δ=2 boundary condition on the scalar. Only the scalar contribution is affected, and one finds

$$-\frac{1}{2}\mathcal{I}(\Delta = 2, 0) = -\frac{1}{2}\mathcal{I}(\Delta = 1, 0) - \frac{\zeta_3}{8\pi^2}$$

- So the final result is

$$F^{(1)} = -\frac{\zeta_3}{8\pi^2}$$

exactly consistent with the non-trivial large N expansion in the critical scalar theory.

The minimal HS theory

- We can repeat the same calculation in the minimal theory, with even spins only, which should be dual to the O(N) vector model.
- Here we find a surprise. The total one loop free energy is *not* zero, but it is equal to

$$F_{\rm min}^{(1)} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}$$

- This is precisely equal to the value of the S^3 free energy of a single real conformal scalar field…Why?

The minimal HS theory

- So far we have always assumed that Newton's constant is given by $G_N^{-1} = cN$. But there can in principle be subleading corrections in the map between G_N and N.
- Because the one-loop piece is precisely proportional to the expected classical piece, this suggests that the result can be consistent with the duality if we assume a shift N->N-1 so that the classical piece is

$$\frac{1}{G_N} F^{(0)}_{\rm min} = (N-1)\left(\frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}\right)$$

which combined with the one-loop piece would give the expected result for F.

One-loop shift

- This effect may perhaps be thought as a finite “one-loop renormalization” of the bare coupling constant in Vasiliev’s theory, somewhat similar to the one-loop shift of the level in Chern-Simons gauge theory
- The fact that the shift is simply an integer is consistent with the idea that the coupling constant in Vasiliev’s theory should be quantized (an argument for this was given by Maldacena-Zhiboedov based on higher spin symmetry).
- This shift will also affect correlation functions. For instance, the graviton 2-point function should receive non-trivial one-loop corrections (proportional to the classical result) in the minimal theory, but not in the “non-minimal” one. Would be interesting (but quite non-trivial) to check.

One-loop shift

- One can also identify a Vasiliev's theory whose spectrum is dual to singlet operators of a Usp(N) vector model (SG, Klebanov): here the spectrum contains one of each even spin, and three of each odd spin. In this case, the bulk one-loop calculation suggests an analogous shift with opposite sign N->N+1
- Interestingly, exactly the same shifts by +/- 1 seem to appear in the map between topological strings and pure Chern-Simons theory with O(N)/Usp(N) gauge groups (Vafa, Sinha).

General dimensions

- There is a formulation of Vasiliev's theory in arbitrary dimensions. The equations of motion have a AdS_{d+1} vacuum solution, and the linearized spectrum around this background is

$$\text{Spectrum}: \qquad s = 1, 2, 3, \ldots, \infty \qquad \text{gauge fields}$$
$$s = 0, \quad m^2 = -2(d-2) \quad \text{scalar}$$

- This spectrum is in one-to-one correspondence with the single trace primaries of a free scalar vector model in dimension d (the scalar bilinear has dimension Δ=d-2). Above d=3, there are no interacting IR fixed points dual to alternate boundary conditions. (But there is a UV fixed point in d=5 of the scalar theory with ϕ^4 interaction).

General dimensions

- It is then natural to conjecture that the singlet sector of the free vector model in dimension d is dual to the Vasiliev's theory in AdS_{d+1} in general d.
- The results for the spectral zeta-function of HS fields in AdS was obtained by Camporesi and Higuchi for general dimensions. It is natural to repeat the one-loop calculations we have done in general dimensions
- This will also give us a more general perspective on how to choose a consistent regularization for the sum over spins

Regulating the sum over spins

- In the AdS_4 case, we computed $\zeta'_{(\Delta,s)}(0)$ and $\zeta_{(\Delta,s)}(0)$ separately for each spin, and then summed over spins assuming the usual Riemann zeta-function prescription
- In general dimensions, this approach does not appear to work.
- Need a more systematic way to regularize the sum

Regulating the sum over spins

- The spectral zeta-function in fact suggests a rather natural regulator

$$\zeta_{(\Delta,s)}(z) \sim \int du \frac{\mu_s(u)}{[u^2 + (\Delta - d/2)^2]^z}$$

- In standard applications, the spectral parameter is introduced to analytically regularize the sum (or integral) over eigenvalues.
- But in our case we see that we can use the spectral parameter z to also regulate the sum over spins! The u-integral and s-sum are convergent for sufficiently large z, and the result can be analytically continued in z.

Regulating the sum over spins

- We may first sum over all spins at arbitrary spectral parameter z, and so introduce a “higher spin spectral function”

$$\zeta_{HS}(z) = \zeta_{(d-2,0)}(z) + \sum_{s=1}^{\infty} \left[\zeta_{(s+d-2,s)}(z) - \zeta_{(s+d-1,s-1)}(z)\right]$$

- Then the regularized one-loop free energy is given by

$$F^{(1)} = -\frac{1}{2}\zeta'_{HS}(0) - \frac{1}{2}\zeta_{HS}(0)\log(\ell^2\Lambda^2)$$

- This regulator seems natural, unambiguous, and dimension independent

Regulating the sum over spins

- A technical remark: in practice, this regulator turns out to be equivalent to computing the determinants for each spin first, as we did before, and then regulate the sum over spins with a generalized (dimension dependent) zeta-function regulator

$$\zeta_{\rm HS}(0) = \zeta_{(d-2,0)}(0) + \lim_{\alpha\to 0}\sum_{s=1}^{\infty}\left(s+\frac{d-3}{2}\right)^{-\alpha}\left(\zeta_{(d+s-2,s)}(0) - \zeta_{(d+s-1,s-1)}(0)\right)$$

and similarly for $\zeta'(0)$.

- For d=3 (AdS_4) this coincides with the simplest Riemann zeta regulator we had assumed earlier.

Results: odd boundary dim.

- There are two physically distinct situations: the case of odd d (even dim. AdS_{d+1}), or even d (odd dim. AdS_{d+1})
- In the odd d case, the situation is similar to the AdS_4 case described earlier. The bulk free energy on Euclidean AdS_{d+1} is dual to the CFT free energy on S^d, which is some finite number for d odd. For free CFT, the large N expansion should be trivial, so we would expect a trivial one-loop bulk free energy.

Results: odd boundary dim.

- For any odd d, the volume of AdS_{d+1} can be regulated into a finite number

$$\int \mathrm{vol}_{\mathrm{AdS}_{d+1}} = \pi^{d/2}\Gamma\left(-\tfrac{d}{2}\right)$$

- But, as in AdS_4, there is in principle a UV logarithmic divergence in the bulk (conformal anomaly) that should vanish for the correspondence to work
- Using the regulator defined above, we find that the UV divergence indeed *vanishes in any odd d*, both for minimal and non-minimal theories. Vasiliev theory is one-loop UV finite in any dimension.

Results: odd boundary dim.

- For the non-minimal theory (s=0,1,2,..), in all d we find

$$F^{(1)} = 0$$

- For the minimal theory (s=0,2,4,...), in all d we find again the curious (and quite non-trivial!) identity

$$F^{(1)}_{\text{min HS}} = F^{\text{conf. scalar}}_{S^d}$$

- As in AdS_4, this can be consistent with the duality if we assume a non-trivial shift $G_N \sim \frac{1}{N-1}$ in the minimal theory.

Results: even boundary dim.

- In even d, the CFT sphere free energy is UV logarithmically divergent, the coefficient being related to the *a-anomaly.*
- In the bulk, this logarithmic divergence is reflected in the divergence of the AdS_{d+1} volume for odd d+1

$$\int \mathrm{vol}_{\mathrm{AdS}_{d+1}} = \frac{2(-\pi)^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \log R$$

- The coefficient of logR in the bulk free energy is dual to the a-anomaly coefficient on the CFT side
- There is no UV divergence in the bulk in this case (in odd dimensional spacetime, $\zeta(0)=0$ identically)

Results: even boundary dim.

- Again, if the CFT is free, the a-anomaly should be Na_{scalar}, without 1/N corrections.
- The results we find are again consistent with the general picture:

$$F^{(1)} = 0$$

$$F^{(1)}_{\text{min HS}} = F^{\text{conf. scalar}}_{S^d} = a_{\text{scalar}} \log R$$

where a_{scalar} is the a-anomaly coefficient of one real conformal scalar in d-dimensions (e.g. a_{scalar}=1/90, -1/756, 23/113400… in d=4,6,8…).

Example: AdS_5

- For example, for the Vasiliev theory in AdS_5 with all integer spins, we find the one-loop bulk free energy

$$\begin{aligned} F^{(1)} &= -\frac{\log R}{360}\sum_{s=1}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right) \\ &= -\left(\frac{1}{18}\zeta(-3)+\frac{7}{60}\zeta(-5)\right)\log R = 0 \end{aligned}$$

- And for the minimal theory with even spins only

$$\begin{aligned} F^{(1)}_{\text{min HS}} &= -\frac{\log R}{360}\sum_{s=2,4,\ldots}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right) \\ &= -\left(\frac{4}{9}\zeta(-3)+\frac{56}{15}\zeta(-5)\right)\log R = +\frac{1}{90}\log R \end{aligned}$$

The number +1/90 is precisely the a-anomaly coefficient of a real scalar in 4d. This fits our general N->N-1 one-loop shift.

Conclusion and summary

- We have obtained new tests of higher spin/vector model dualities, by computing the free energy on the sphere on both sides.
- The classical bulk contribution is still out of reach (lacking understanding of the Lagrangian), but the one-loop calculation is well defined and can be done explicitly in general dimensions.
- In all dimensions, we find that one-loop UV divergences in the Vasiliev theory vanish due to the contribution of the infinite tower of spins. A simple but non-trivial consistency test that these can be "UV complete" AdS/CFT dualities.

Conclusion and summary

- In the “non-minimal” theories with all integer spins, the bulk one-loop free energy exactly vanishes, in agreement with the absence of $O(N^0)$ corrections in the dual free vector model.
- In the “minimal” theories with all even spins, the bulk one-loop free energy in AdS_{d+1} is exactly equal to the free energy of a real scalar field on S^d, suggesting that in all dimensions the duality requires a shift $G_N \sim 1/(N-1)$ in the identification of the bulk coupling constant. It would be interesting to test this further for instance by computing loop corrections to correlation functions.