Entanglement Entropy and RG Flow

Tatsuma Nishioka

(IAS)

1204.4160, 1207.3360
with I. Klebanov, S. Pufu, B. Safdi (Princeton)

1301.0336 with C. Herzog (Stony Brook)

1306.2958 with I. Yaakov (Princeton)

Motivation

- Entanglement entropy as a measure of degrees of freedom
  - F-theorem in three-dimensions
- An order parameter for various phase transitions
- Difficulties in analytical computations
  - QFT on a singular space
  - IR divergences

Motivation

- Entanglement entropy as a measure of degrees of freedom
  - F-theorem in three-dimensions

- An order parameter for various phase transitions

- Difficulties in analytical computations
  - QFT on a singular space
  - IR divergences

Motivation

- Entanglement entropy as a measure of degrees of freedom
  - F-theorem in three-dimensions
- An order parameter for various phase transitions
- Difficulties in analytical computations
  - QFT on a singular space
  - IR divergences

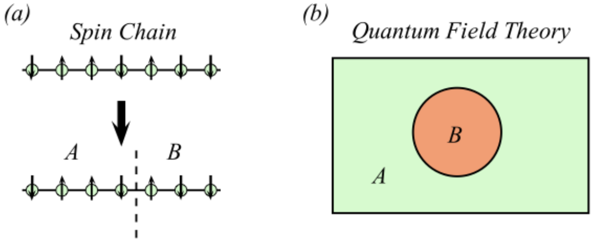
Outline

Outline

1. Review of Entanglement Entropy
   - Replica trick
   - Strong subadditivity
2. Entanglement Entropy in Two-dimensions
3. Entanglement Entropy in Three-dimensions
4. Supersymmetric Rényi entropy
5. Summary

Definition of entanglement entropy

- Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$

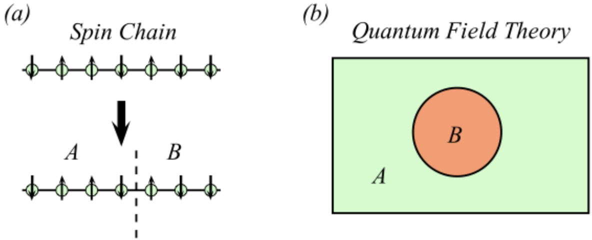


Definition

$$S_A = -\mathrm{tr}_A \rho_A \log \rho_A$$

Definition of entanglement entropy

- Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$



Definition

$$S_A = -\mathrm{tr}_A \rho_A \log \rho_A$$

Definition of entanglement entropy

Definition

$$S_A = -\mathrm{tr}_A \rho_A \log \rho_A$$

- $|\Psi\rangle$: wave function of a ground state

$$\rho_{tot} = \frac{1}{\langle\Psi|\Psi\rangle} |\Psi\rangle\langle\Psi|$$

Definition of entanglement entropy

Definition

$$S_A = -\mathrm{tr}_A \rho_A \log \rho_A$$

- $|\Psi\rangle$: wave function of a ground state

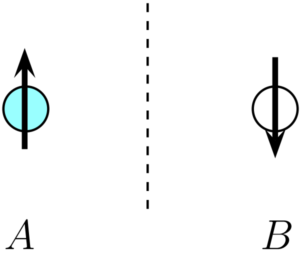
$$\rho_{tot} = \frac{1}{\langle\Psi|\Psi\rangle} |\Psi\rangle\langle\Psi|$$

- Reduced density matrix:

$$\rho_A = \mathrm{tr}_B \rho_{tot} = \sum_i \langle \psi_B^i | \rho_{tot} | \psi_B^i \rangle$$

$\mathcal{H}_B = \{|\psi_B^1\rangle, |\psi_B^2\rangle, \cdots\}$ orthonormal basis

Example: two spin system



- Hilbert spaces: $\mathcal{H}_A = \{|\uparrow\rangle_A, |\downarrow\rangle_A\}$, $\mathcal{H}_B = \{|\uparrow\rangle_B, |\downarrow\rangle_B\}$

Example: two spin system

- Given a ground state ($\langle\Psi|\Psi\rangle = 1$):

$$|\Psi\rangle = \cos\theta\,|\uparrow\rangle_A|\downarrow\rangle_B + \sin\theta\,|\downarrow\rangle_A|\uparrow\rangle_B$$

Example: two spin system

- Given a ground state ($\langle\Psi|\Psi\rangle = 1$):

$$|\Psi\rangle = \cos\theta\,|\uparrow\rangle_A|\downarrow\rangle_B + \sin\theta\,|\downarrow\rangle_A|\uparrow\rangle_B$$

- Reduce density matrix:

$$\begin{aligned}\rho_A &= {}_B\langle\downarrow|\Psi\rangle\langle\Psi|\downarrow\rangle_B + {}_B\langle\uparrow|\Psi\rangle\langle\Psi|\uparrow\rangle_B \\ &= \cos^2\theta|\uparrow\rangle_A\,{}_A\langle\uparrow| + \sin^2\theta|\downarrow\rangle_A\,{}_A\langle\downarrow|\end{aligned}$$

Example: two spin system

- Reduce density matrix:

$$\begin{aligned}\rho_A &= {}_B\langle\downarrow|\Psi\rangle\langle\Psi|\downarrow\rangle_B + {}_B\langle\uparrow|\Psi\rangle\langle\Psi|\uparrow\rangle_B \\ &= \cos^2\theta|\uparrow\rangle_A\,{}_A\langle\uparrow| + \sin^2\theta|\downarrow\rangle_A\,{}_A\langle\downarrow|\end{aligned}$$

- Matrix notation:

$$\rho_A = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Example: two spin system

- Matrix notation:

$$\rho_A = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

- EE as a function of θ

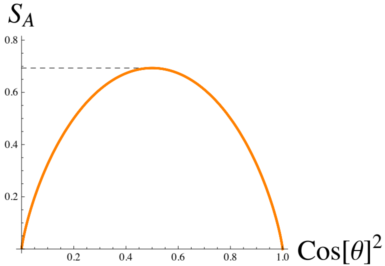
$$\begin{aligned} S_A &= -\mathrm{tr}_A \rho_A \log \rho_A \\ &= -\cos^2\theta \log(\cos^2\theta) - \sin^2\theta \log(\sin^2\theta) \end{aligned}$$

Example: two spin system

- EE as a function of θ

$$\begin{aligned} S_A &= -\mathrm{tr}_A \rho_A \log \rho_A \\ &= -\cos^2\theta \log(\cos^2\theta) - \sin^2\theta \log(\sin^2\theta) \end{aligned}$$

  - $\cos^2\theta = \frac{1}{2}$: Maximally entangled, $S_A = \log 2$
  - $\cos^2\theta = 0, 1$: No entanglement , $S_A = 0$



QFTs and replica trick

- Not easy to compute ρ_A in QFT
- Useful trick:

$$S_A = -\partial_n \log \mathrm{tr}_A \rho_A^n \big|_{n=1} \qquad (\mathrm{tr}_A \rho_A = 1)$$

- Z_n: partition function on n-covering space

$$\mathrm{tr}_A \rho_A^n = \frac{Z_n}{(Z_1)^n}$$

QFTs and replica trick

- Not easy to compute ρ_A in QFT
- Useful trick:

$$S_A = -\partial_n \log \mathrm{tr}_A \rho_A^n \Big|_{n=1} \qquad (\mathrm{tr}_A \rho_A = 1)$$

- Z_n: partition function on n-covering space

$$\mathrm{tr}_A \rho_A^n = \frac{Z_n}{(Z_1)^n}$$

QFTs and replica trick

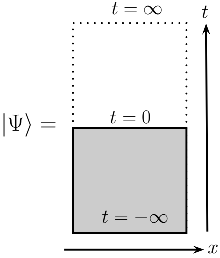
- Not easy to compute ρ_A in QFT
- Useful trick:

$$S_A = -\partial_n \log \mathrm{tr}_A \rho_A^n \Big|_{n=1} \qquad (\mathrm{tr}_A \rho_A = 1)$$

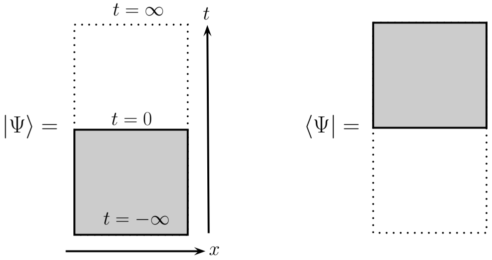
- Z_n: partition function on n-covering space

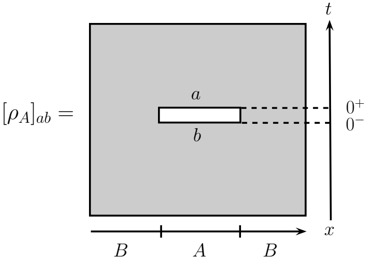
$$\mathrm{tr}_A \rho_A^n = \frac{Z_n}{(Z_1)^n}$$

Replica trick and covering space



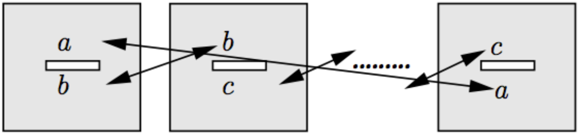
Replica trick and covering space





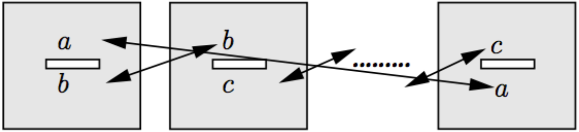
Replica trick and covering space

$\mathrm{Tr}(\rho_A)^n =$



Replica trick and covering space

$$\mathrm{Tr}(\rho_A)^n =$$



Entanglement entropy

$$S_A = -\left(\partial_n - 1\right)\log Z_n\Big|_{n=1}$$

Properties of entanglement entropy

- At zero temperature, for pure ground state

$$S_A = S_B$$

- Strong subadditivity [Lieb-Ruskai '73]:

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}$$
$$S_A + S_C \leq S_{A+B} + S_{B+C}$$

for any three disjoint regions A, B and C

Properties of entanglement entropy

- At zero temperature, for pure ground state

$$S_A = S_B$$

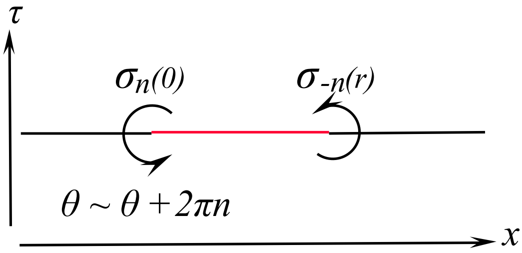
- Strong subadditivity [Lieb-Ruskai '73]:

$$\begin{aligned} S_{A+B+C} + S_B &\le S_{A+B} + S_{B+C} \\ S_A + S_C &\le S_{A+B} + S_{B+C} \end{aligned}$$

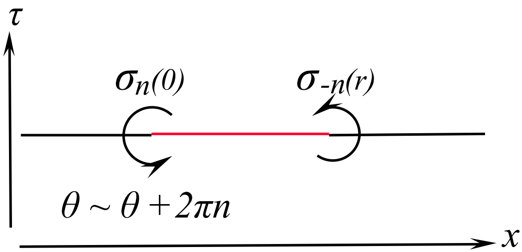
for any three disjoint regions A, B and C

Outline

Entanglement entropy in CFT_2



Entanglement entropy in CFT_2



- CFT_2: (σ_n : twist fields of $\Delta_n = \bar{\Delta}_n = c(n - \frac{1}{n})/24$)

$$Z_n = \langle \sigma_n(0)\sigma_{-n}(r)\rangle = \left(\frac{r}{\epsilon}\right)^{-\frac{c}{6}(n-\frac{1}{n})}$$

Entanglement entropy in CFT_2

- CFT_2: (σ_n : twist fields of $\Delta_n = \bar{\Delta}_n = c(n - \frac{1}{n})/24$)

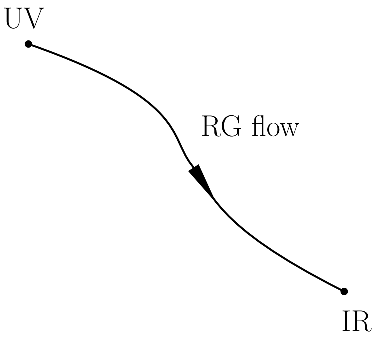
$$Z_n = \langle \sigma_n(0)\sigma_{-n}(r)\rangle = \left(\frac{r}{\epsilon}\right)^{-\frac{c}{6}(n-\frac{1}{n})}$$

- Rényi entropy: (EE in $n \to 1$ limit)

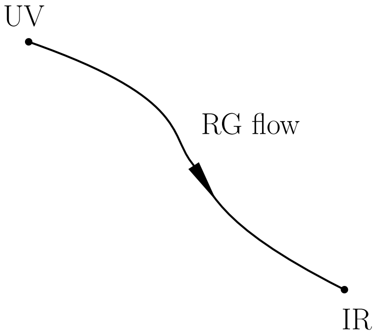
$$S_n = \frac{\log Z_n - n\log Z_1}{1-n} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{r}{\epsilon}\right)$$
$$\to \frac{c}{3}\log\left(\frac{r}{\epsilon}\right) \quad (n\to 1)$$

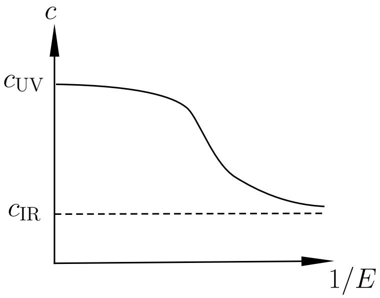
c: central charge

RG flow and c-function



RG flow and c-function





Entropic c-theorem

- 2d entropic c-function:

$$c(r) \equiv 3r\frac{dS_A(r)}{dr}$$

- Interpolate two fixed points

$$c(r) \to c_{\rm UV} \quad (r \to 0) \ , \qquad c(r) \to c_{\rm IR} \quad (r \to \infty)$$

- SSA + Lorentz invariance ⇒ monotonicity [Casini-Huerta 04]

$$c'(r) \le 0$$

Entropic c-theorem

- 2d entropic c-function:

$$c(r) \equiv 3r \frac{dS_A(r)}{dr}$$

- Interpolate two fixed points

$$c(r) \to c_{\mathrm{UV}} \quad (r \to 0) \ , \qquad c(r) \to c_{\mathrm{IR}} \quad (r \to \infty)$$

- SSA + Lorentz invariance ⇒ monotonicity [Casini-Huerta 04]

$$c'(r) \le 0$$

Entropic c-theorem

- 2d entropic c-function:

$$c(r) \equiv 3r \frac{dS_A(r)}{dr}$$

- Interpolate two fixed points

$$c(r) \to c_{\text{UV}} \quad (r \to 0) \ , \qquad c(r) \to c_{\text{IR}} \quad (r \to \infty)$$

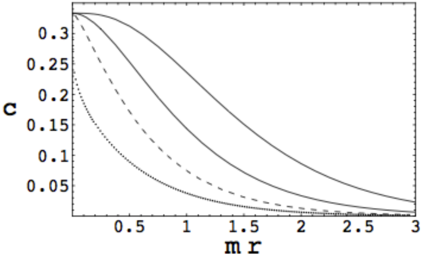
- SSA + Lorentz invariance $\Rightarrow$ monotonicity [Casini-Huerta 04]

$$c'(r) \le 0$$

EE of free massive fields

- Massless scalar field and Dirac fermion: $c = 1$
- Not stationary at UV fixed point [Casini-Huerta 06]

$$c_s(t) = 1 + \frac{3}{2\log t} + \cdots$$
$$c_D(t) = 1 - t^2 \log^2 t + \cdots$$
$$(t = mr)$$

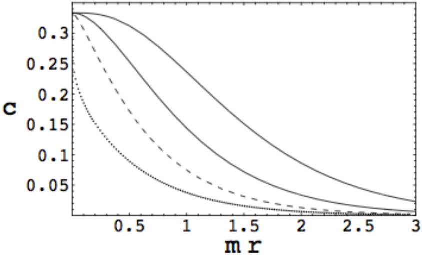


- Different from the Zamolodchikov's c-function

EE of free massive fields

- Massless scalar field and Dirac fermion: $c = 1$
- Not stationary at UV fixed point [Casini-Huerta 06]

$$c_s(t) = 1 + \frac{3}{2\log t} + \cdots$$
$$c_D(t) = 1 - t^2 \log^2 t + \cdots$$
$$(t = mr)$$

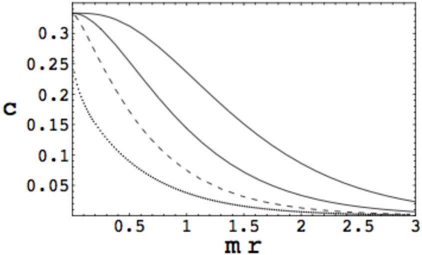


- Different from
  the Zamolodchikov's c-function

EE of free massive fields

- Massless scalar field and Dirac fermion: $c = 1$
- Not stationary at UV fixed point [Casini-Huerta 06]

$$c_s(t) = 1 + \frac{3}{2\log t} + \cdots$$
$$c_D(t) = 1 - t^2 \log^2 t + \cdots$$
$$(t = mr)$$



- Different from the Zamolodchikov's c-function

Massive Dirac fermion and sine-Gordon model

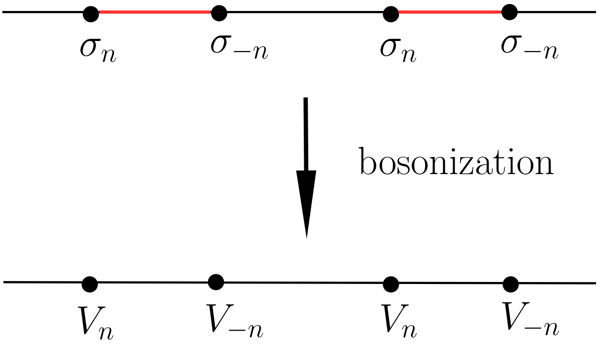
Dirac fermion	sine-Gordon
fermion: ψ	scalar: ϕ

Massive Dirac fermion and sine-Gordon model

Dirac fermion	sine-Gordon
fermion: ψ	scalar: ϕ
mass: $m\bar{\psi}\psi$	potential: $\lambda \cos\phi$

Massive Dirac fermion and sine-Gordon model

Dirac fermion	sine-Gordon
fermion: ψ	scalar: ϕ
mass: $m\bar{\psi}\psi$	potential: $\lambda\cos\phi$
twist op: σ_n	vertex op: $V_n \sim e^{i\frac{\phi}{n}}$



Mass expansions

- One interval ($\lambda \propto m$):

$$Z_n = \langle V_n(0) V_{-n}(r) \rangle$$

Mass expansions

- One interval ($\lambda \propto m$):

$$
\begin{aligned}
Z_n = & \langle V_n(0) V_{-n}(r) \rangle \\
& + \frac{\lambda^2}{2} \int d^2x d^2y \, \langle \cos\phi(x) \cos\phi(y) V_n(0) V_{-n}(r) \rangle \\
& + \mathcal{O}(\lambda^4)
\end{aligned}
$$

Mass expansions

- One interval ($\lambda \propto m$):

$$Z_n = \langle V_n(0) V_{-n}(r) \rangle + \frac{\lambda^2}{2} \int d^2x d^2y \, \langle \cos\phi(x) \cos\phi(y) V_n(0) V_{-n}(r) \rangle + \mathcal{O}(\lambda^4)$$

- Higher order terms are given by correlation functions of the vertex operators

IR divergence in EE

- EE of a massive Dirac fermion on a torus of size $\Lambda \gg r$ [Herzog-TN 13]

$$c_D(t) = 1 - t^2 \log^2 \Lambda + \cdots$$

- The IR cutoff Λ would be identified with $1/t = 1/(mr)$
- Reproduce the result in the flat space limit
- Perturbative expansion may be possible on a compact space

IR divergence in EE

- EE of a massive Dirac fermion on a torus of size $\Lambda \gg r$ [Herzog-TN 13]

$$c_D(t) = 1 - t^2 \log^2 \Lambda + \cdots$$

- The IR cutoff Λ would be identified with $1/t = 1/(mr)$
- Reproduce the result in the flat space limit
- Perturbative expansion may be possible on a compact space

IR divergence in EE

- EE of a massive Dirac fermion on a torus of size $\Lambda \gg r$ [Herzog-TN 13]

$$c_D(t) = 1 - t^2 \log^2 \Lambda + \cdots$$

- The IR cutoff Λ would be identified with $1/t = 1/(mr)$
- Reproduce the result in the flat space limit
- Perturbative expansion may be possible on a compact space

IR divergence in EE

- EE of a massive Dirac fermion on a torus of size $\Lambda \gg r$ [Herzog-TN 13]

$$c_D(t) = 1 - t^2 \log^2 \Lambda + \cdots$$

- The IR cutoff Λ would be identified with $1/t = 1/(mr)$
- Reproduce the result in the flat space limit
- Perturbative expansion may be possible on a compact space

Entanglement and c-theorem in 2d

- Entropic c-function (not stationary at a fixed point)

$$c(t) = c \quad \text{for CFT} \ , \qquad c'(t) \leq 0$$

- Zamolodchikov's c-function (stationary at a fixed point)

$$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \leq 0 \ , \qquad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- Thermal c-function

$$F_{\text{Therm}} \sim c T^2$$

- Every c-function coincides at a conformal fixed point

Entanglement and c-theorem in 2d

- Entropic c-function (not stationary at a fixed point)

$$c(t) = c \quad \text{for CFT} \ , \qquad c'(t) \le 0$$

- Zamolodchikov's c-function (stationary at a fixed point)

$$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \le 0 \ , \qquad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- Thermal c-function

$$F_{\text{Therm}} \sim c T^2$$

- Every c-function coincides at a conformal fixed point

Entanglement and c-theorem in 2d

- Entropic c-function (not stationary at a fixed point)

$$c(t) = c \quad \text{for CFT} \ , \qquad c'(t) \le 0$$

- Zamolodchikov's c-function (stationary at a fixed point)

$$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \le 0 \ , \qquad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- Thermal c-function

$$F_{\text{Therm}} \sim c\, T^2$$

- Every c-function coincides at a conformal fixed point

Entanglement and c-theorem in 2d

- Entropic c-function (not stationary at a fixed point)

$$c(t) = c \quad \text{for CFT} \ , \qquad c'(t) \le 0$$

- Zamolodchikov's c-function (stationary at a fixed point)

$$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \le 0 \ , \qquad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- Thermal c-function

$$F_{\text{Therm}} \sim c\, T^2$$

- Every c-function coincides at a conformal fixed point

Outline

1. Review of Entanglement Entropy
2. Entanglement Entropy in Two-dimensions
3. Entanglement Entropy in Three-dimensions
   - Renormalized EE and F-theorem
   - REE of free massive fields
4. Supersymmetric Rényi entropy
5. Summary

C-theorem in 3d?

- Thermal c-theorem:

$$F_{\text{Therm}} \sim c_{\text{Therm}} T^3$$

- C_T-theorem: [Petkou 94]

$$C_T|_{UV} \geq C_T|_{IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

- F-theorem: [Jafferis-Klebanov-Pufu-Safdi 11, Klebanov-Pufu-Safdi 11]

$$F_{\text{UV}}(S^3) \geq F_{\text{IR}}(S^3) \ , \quad F = -\log Z(S^3)$$

C-theorem in 3d?

- Thermal c-theorem: Counter example by [Sachdev 93]

$$F_{\text{Therm}} \sim c_{\text{Therm}} T^3$$

- C_T-theorem: [Petkou 94]

$$C_T|_{UV} \geq C_T|_{IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

- F-theorem: [Jafferis-Klebanov-Pufu-Safdi 11, Klebanov-Pufu-Safdi 11]

$$F_{\text{UV}}(S^3) \geq F_{\text{IR}}(S^3) \ , \quad F = -\log Z(S^3)$$

C-theorem in 3d?

- Thermal c-theorem: Counter example by [Sachdev 93]

$$F_{\text{Therm}} \sim c_{\text{Therm}} T^3$$

- C_T-theorem: [Petkou 94]

$$C_T|_{UV} \geq C_T|_{IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

- F-theorem: [Jafferis-Klebanov-Pufu-Safdi 11, Klebanov-Pufu-Safdi 11]

$$F_{\text{UV}}(S^3) \geq F_{\text{IR}}(S^3) \ , \quad F = -\log Z(S^3)$$

C-theorem in 3d?

- Thermal c-theorem: Counter example by [Sachdev 93]

$$F_{\text{Therm}} \sim c_{\text{Therm}} T^3$$

- C_T-theorem: [Petkou 94] Counter example by [TN-Yonekura 13]

$$C_T|_{UV} \geq C_T|_{IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

- F-theorem: [Jafferis-Klebanov-Pufu-Safdi 11, Klebanov-Pufu-Safdi 11]

$$F_{\text{UV}}(S^3) \geq F_{\text{IR}}(S^3) \ , \quad F = -\log Z(S^3)$$

C-theorem in 3d?

- Thermal c-theorem: Counter example by [Sachdev 93]

$$F_{\text{Therm}} \sim c_{\text{Therm}} T^3$$

- C_T-theorem: [Petkou 94] Counter example by [TN-Yonekura 13]

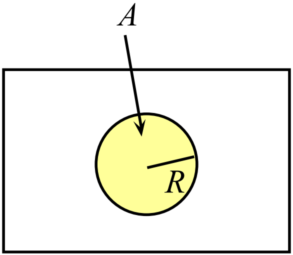
$$C_T|_{UV} \geq C_T|_{IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

- F-theorem: [Jafferis-Klebanov-Pufu-Safdi 11, Klebanov-Pufu-Safdi 11]

$$F_{\text{UV}}(S^3) \geq F_{\text{IR}}(S^3) \ , \quad F = -\log Z(S^3)$$

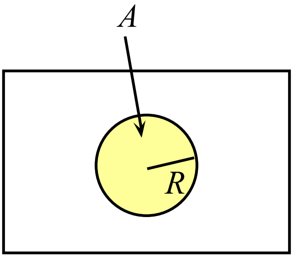
EE in CFT_3 and F-theorem

$\partial A = S^1$ of radius R in CFT_3



EE in CFT$_3$ and F-theorem

$\partial A = S^1$ of radius R in CFT$_3$



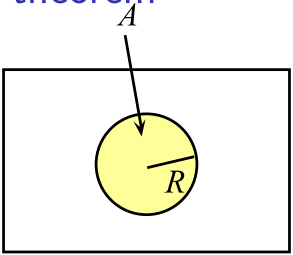
For CFT$_3$ [Casini-Huerta-Myers 11]

$$S_A(R) = -F(S^3)$$

up to a UV divergence

EE in CFT$_3$ and F-theorem

$\partial A = S^1$ of radius R in CFT$_3$



For CFT$_3$ [Casini-Huerta-Myers 11]

$$S_A(R) = -F(S^3)$$

up to a UV divergence

Proof of F-theorem by using entanglement entropy?

Renormalized entanglement entropy

- Interpolating function between $F_{\rm UV}$ and $F_{\rm IR}$

Renormalized entanglement entropy

- Interpolating function between $F_{\rm UV}$ and $F_{\rm IR}$
- Monotonically decreasing under RG flow

Renormalized entanglement entropy

- Interpolating function between $F_{\rm UV}$ and $F_{\rm IR}$
- Monotonically decreasing under RG flow

Renormalized entanglement entropy [Liu-Mezei 12]

$$\mathcal{F}(R) \equiv (R\partial_R - 1) S_A(R)$$

Renormalized entanglement entropy

Renormalized entanglement entropy [Liu-Mezei 12]

$$\mathcal{F}(R) \equiv (R\partial_R - 1) S_A(R)$$

- For CFT$_3$

$$S_A(R) = \alpha \frac{2\pi R}{\epsilon} - F(S^3) \qquad \Rightarrow \qquad \mathcal{F}(R) = F(S^3)$$

Renormalized entanglement entropy

Renormalized entanglement entropy [Liu-Mezei 12]

$$\mathcal{F}(R) \equiv (R\partial_R - 1) S_A(R)$$

- For CFT$_3$

$$S_A(R) = \alpha \frac{2\pi R}{\epsilon} - F(S^3) \qquad \Rightarrow \qquad \mathcal{F}(R) = F(S^3)$$

- Proof of monotonicity [Casini-Huerta 12]

$$\text{SSA} + \text{Lorentz invariance} \qquad \Rightarrow \qquad \mathcal{F}'(R) = R\, S''(R) \leq 0$$

EE in gapped phase

- Large m expansion: [cf. Grover-Turner-Vishwanath 11]

$$S_A(R) = \alpha \frac{\ell_\Sigma}{\epsilon} + \beta\, m\, \ell_\Sigma - \gamma + \sum_{l=0}^{\infty} \frac{c^\Sigma_{-1-2l}}{m^{2l+1}}$$

  - ℓ_Σ: length of $\Sigma = \partial A$
  - γ: topological entanglement entropy [Kitaev-Preskill 05, Levin-Wen 05]

- Dimensional reduction: $\mathbb{R}^{2,1} \times S^1 \to \mathbb{R}^{2,1}$ [Huerta 11, Klebanov-TN-Pufu-Safdi 12]

- Entangling surface: $\Sigma \times S^1 \to \Sigma$

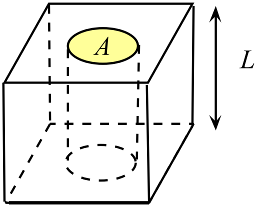
EE in gapped phase

- Large m expansion: [cf. Grover-Turner-Vishwanath 11]

$$S_A(R) = \alpha \frac{\ell_\Sigma}{\epsilon} + \beta\, m\, \ell_\Sigma - \gamma + \sum_{l=0}^{\infty} \frac{c^\Sigma_{-1-2l}}{m^{2l+1}}$$

  - ℓ_Σ: length of $\Sigma = \partial A$
  - γ: topological entanglement entropy [Kitaev-Preskill 05, Levin-Wen 05]

- Dimensional reduction: $\mathbb{R}^{2,1} \times S^1 \to \mathbb{R}^{2,1}$ [Huerta 11, Klebanov-TN-Pufu-Safdi 12]



- Entangling surface: $\Sigma \times S^1 \to \Sigma$

EE in gapped phase

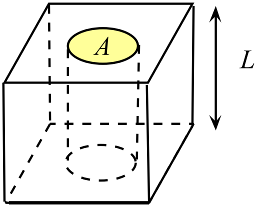
- Large m expansion: [cf. Grover-Turner-Vishwanath 11]

$$S_A(R) = \alpha \frac{\ell_\Sigma}{\epsilon} + \beta\, m\, \ell_\Sigma - \gamma + \sum_{l=0}^{\infty} \frac{c^{\Sigma}_{-1-2l}}{m^{2l+1}}$$

  - ℓ_Σ: length of $\Sigma = \partial A$
  - γ: topological entanglement entropy [Kitaev-Preskill 05, Levin-Wen 05]

- Dimensional reduction: $\mathbb{R}^{2,1} \times S^1 \to \mathbb{R}^{2,1}$ [Huerta 11, Klebanov-TN-Pufu-Safdi 12]

- Entangling surface: $\Sigma \times S^1 \to \Sigma$



Dimensional reduction

- 4d EE from 3d EE

$$S^{(3+1)}_{\Sigma\times S^1} = \sum_{n\in\mathbb{Z}} S^{(2+1)}_{\Sigma}\left(m = \left|\frac{2\pi n}{L}\right|\right)$$

Dimensional reduction

- 4d EE from 3d EE

$$S^{(3+1)}_{\Sigma\times S^1} = \sum_{n\in\mathbb{Z}} S^{(2+1)}_{\Sigma}\left(m = \left|\frac{2\pi n}{L}\right|\right)$$

- $L \to \infty$ limit:

$$S^{(3+1)}_{\Sigma\times S^1} \sim \int^{1/\epsilon} dp\, S^{(2+1)}_{\Sigma}(p)$$

Dimensional reduction

- $L \to \infty$ limit:

$$S^{(3+1)}_{\Sigma\times S^1} \sim \int^{1/\epsilon} dp\, S^{(2+1)}_{\Sigma}(p)$$

$$\cup \qquad\qquad\qquad \cup$$

$$c_0^{(3+1)} \log \epsilon \qquad\qquad c_{-1}^{\Sigma} \log(1/\epsilon)$$

Dimensional reduction

- $L \to \infty$ limit:

$$S^{(3+1)}_{\Sigma\times S^1} \sim \int^{1/\epsilon} dp\, S^{(2+1)}_{\Sigma}(p)$$

$$\cup \qquad\qquad\qquad \cup$$

$$c_0^{(3+1)} \log\epsilon \qquad\qquad c_{-1}^{\Sigma} \log(1/\epsilon)$$

$$\boxed{c_{-1}^{\Sigma} = \frac{1}{480}(n_0 + 3n_{1/2}) \oint_{\Sigma} ds\, \kappa^2}$$

(n_0: # of scalars, $n_{1/2}$: # of fermions)

Renormalized EE in IR region

- REE: $\Sigma =$ a circle of radius $R \quad \Rightarrow \quad \kappa = \frac{1}{R}$

Renormalized EE in IR region

- REE: $\Sigma =$ a circle of radius $R \quad \Rightarrow \quad \kappa = \frac{1}{R}$

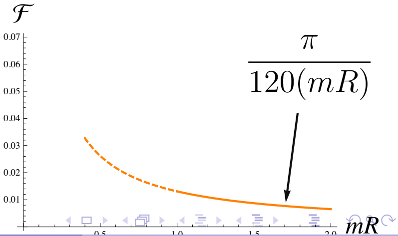
$$\mathcal{F}(R) = -\frac{2c_{-1}^{\Sigma}}{m} + \mathcal{O}\left(\frac{1}{(mR)^3}\right)$$
$$c_{-1}^{\Sigma} = -\frac{\pi}{240R}$$

Renormalized EE in IR region

- REE: Σ = a circle of radius R $\quad\Rightarrow\quad$ $\kappa = \frac{1}{R}$

$$\mathcal{F}(R) = -\frac{2c_{-1}^{\Sigma}}{m} + \mathcal{O}\left(\frac{1}{(mR)^3}\right)$$
$$c_{-1}^{\Sigma} = -\frac{\pi}{240R}$$

- REE is monotonically decreasing to zero in IR!



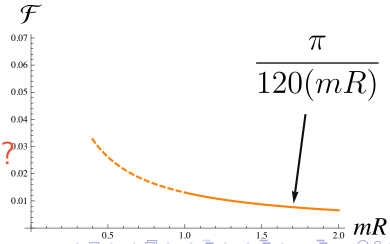
Renormalized EE in IR region

- REE: $\Sigma =$ a circle of radius $R \quad \Rightarrow \quad \kappa = \frac{1}{R}$

$$\mathcal{F}(R) = -\frac{2c^{\Sigma}_{-1}}{m} + \mathcal{O}\left(\frac{1}{(mR)^3}\right)$$
$$c^{\Sigma}_{-1} = -\frac{\pi}{240R}$$

- REE is monotonically decreasing to zero in IR!

- What happens in small mass region?



Numerical study of REE for free massive scalar

[Klebanov-TN-Pufu-Safdi 12]

- Perturbation around $m = 0$ doesn't work
- Numerical method [Huerta 11]: $\mathcal{F}(0) \simeq 0.0638 = F_{UV}(S^3)$
- $\mathcal{F}$ is not stationary at UV fixed point!

$$\partial_{(mR)^2}\mathcal{F} \neq 0$$

- IR divergence?

Numerical study of REE for free massive scalar

[Klebanov-TN-Pufu-Safdi 12]

- Perturbation around $m = 0$ doesn't work
- Numerical method [Huerta 11]: $\mathcal{F}(0) \simeq 0.0638 = F_{\mathrm{UV}}(S^3)$
- $\mathcal{F}$ is not stationary at UV fixed point!

$$\partial_{(mR)^2}\mathcal{F} \neq 0$$

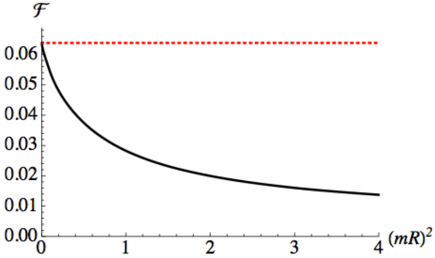
- IR divergence?

Numerical study of REE for free massive scalar

[Klebanov-TN-Pufu-Safdi 12]

- Perturbation around $m = 0$ doesn't work
- Numerical method [Huerta 11]: $\mathcal{F}(0) \simeq 0.0638 = F_{\rm UV}(S^3)$
- $\mathcal{F}$ is not stationary at UV fixed point!

$$\partial_{(mR)^2}\mathcal{F} \neq 0$$



- IR divergence?

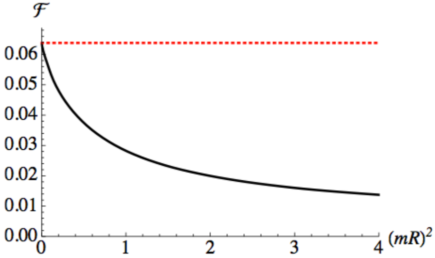
Numerical study of REE for free massive scalar

[Klebanov-TN-Pufu-Safdi 12]

- Perturbation around $m = 0$ doesn't work
- Numerical method [Huerta 11]: $\mathcal{F}(0) \simeq 0.0638 = F_{\rm UV}(S^3)$
- $\mathcal{F}$ is not stationary at UV fixed point!

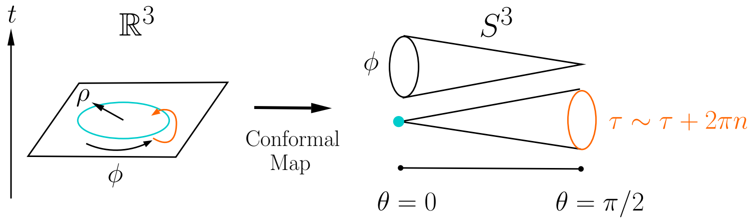
$$\partial_{(mR)^2}\mathcal{F} \neq 0$$

- IR divergence?



Outline

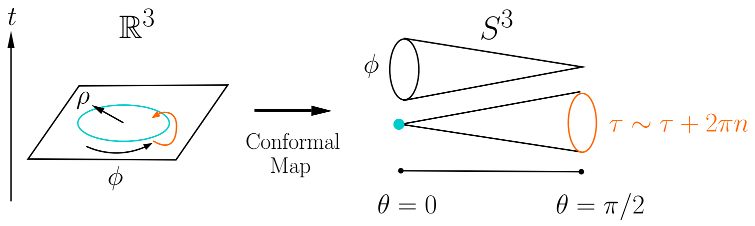
Conformal map



$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\phi^2$$

$$ds^2 = d\theta^2 + \sin^2\theta d\tau^2 + \cos^2\theta d\phi^2$$

Conformal map



$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\phi^2$$

$$ds^2 = d\theta^2 + \sin^2\theta d\tau^2 + \cos^2\theta d\phi^2$$

For CFT

$$Z_n[\mathbb{R}^3] = Z[S_n^3]$$

S_n^3: n-branched covering of S^3

Rényi entropy for CFT

- The Rényi entropy of a disc for CFT

$$S_n = \frac{1}{1-n} \log \frac{Z[S^3_n]}{(Z[S^3])^n}$$

- For free fields, $Z[S^3_n]$: one-loop determinant [Klebanov-Pufu-Sachdev-Safdi 11]
- For SUSY gauge theories, $Z[S^3]$ $(n=1)$ can be obtained by localization [Kapstin-Willet-Yaakov 09, Jafferis, Hama-Hosomichi-Lee 10]

Rényi entropy for CFT

- The Rényi entropy of a disc for CFT

$$S_n = \frac{1}{1-n} \log \frac{Z[S^3_n]}{(Z[S^3])^n}$$

- For free fields, $Z[S^3_n]$: one-loop determinant [Klebanov-Pufu-Sachdev-Safdi 11]
- For SUSY gauge theories, $Z[S^3]$ $(n=1)$ can be obtained by localization [Kapstin-Willet-Yaakov 09, Jafferis, Hama-Hosomichi-Lee 10]

Rényi entropy for CFT

- The Rényi entropy of a disc for CFT

$$S_n = \frac{1}{1-n} \log \frac{Z[S^3_n]}{(Z[S^3])^n}$$

- For free fields, $Z[S^3_n]$: one-loop determinant [Klebanov-Pufu-Sachdev-Safdi 11]
- For SUSY gauge theories, $Z[S^3]$ $(n=1)$ can be obtained by localization [Kapstin-Willet-Yaakov 09, Jafferis, Hama-Hosomichi-Lee 10]

SUSY on singular space

- SUSY is broken on the singular space S^3_n
- To recover SUSY, turn on the R-symmetry flux
- Supersymmetric Rényi entropy [TN-Yaakov 13]

$$S_n^{\text{susy}} = \frac{1}{1-n} \log \left| \frac{Z^{\text{susy}}[S^3_n]}{(Z^{\text{susy}}[S^3])^n} \right|$$

- Exact partition function $Z^{\text{susy}}[S^3_n]$ by localization!

SUSY on singular space

- SUSY is broken on the singular space S_n^3
- To recover SUSY, turn on the R-symmetry flux
- Supersymmetric Rényi entropy [TN-Yaakov 13]

$$S_n^{\text{susy}} = \frac{1}{1-n} \log \left| \frac{Z^{\text{susy}}[S_n^3]}{(Z^{\text{susy}}[S^3])^n} \right|$$

- Exact partition function $Z^{\text{susy}}[S_n^3]$ by localization!

SUSY on singular space

- SUSY is broken on the singular space S_n^3
- To recover SUSY, turn on the R-symmetry flux
- Supersymmetric Rényi entropy [TN-Yaakov 13]

$$S_n^{\text{susy}} = \frac{1}{1-n} \log \left| \frac{Z^{\text{susy}}[S_n^3]}{(Z^{\text{susy}}[S^3])^n} \right|$$

- Exact partition function $Z^{\text{susy}}[S_n^3]$ by localization!

SUSY on singular space

- SUSY is broken on the singular space S^3_n
- To recover SUSY, turn on the R-symmetry flux
- Supersymmetric Rényi entropy [TN-Yaakov 13]

$$S_n^{\text{susy}} = \frac{1}{1-n} \log \left| \frac{Z^{\text{susy}}[S^3_n]}{(Z^{\text{susy}}[S^3])^n} \right|$$

- Exact partition function $Z^{\text{susy}}[S^3_n]$ by localization!

Supersymmetric Rényi entropy

- SRE is not equal to RE due to the R-symmetry flux
- Expansion around $n = 1$:

$$S_n^{\mathrm{susy}} = S_1 + \frac{\pi^2}{16}\tau_{rr}(n-1) + \cdots$$

τ_{rr}: two-point function of the R-currents

- Large-N limit:

$$S_n^{\mathrm{susy}} = \frac{3n+1}{4n} S_1$$

Agrees with the holographic result [Huang-Rey-Zhou 14, TN 14]

Supersymmetric Rényi entropy

- SRE is not equal to RE due to the R-symmetry flux
- Expansion around $n = 1$:

$$S_n^{\text{susy}} = S_1 + \frac{\pi^2}{16}\tau_{rr}(n-1) + \cdots$$

τ_{rr}: two-point function of the R-currents

- Large-N limit:

$$S_n^{\text{susy}} = \frac{3n+1}{4n} S_1$$

Agrees with the holographic result [Huang-Rey-Zhou 14, TN 14]

Supersymmetric Rényi entropy

- SRE is not equal to RE due to the R-symmetry flux

- Expansion around $n = 1$:

$$S_n^{\text{susy}} = S_1 + \frac{\pi^2}{16}\tau_{rr}(n-1) + \cdots$$

τ_{rr}: two-point function of the R-currents

- Large-N limit:

$$S_n^{\text{susy}} = \frac{3n+1}{4n} S_1$$

Agrees with the holographic result [Huang-Rey-Zhou 14, TN 14]

Outline

Summary

- EE a useful measure of degrees of freedom defined in arbitrary dimensions
- Only the F-theorem in three-dimensions
- REE not a c-function in the Zamolodchikov's sense (non-stationarity $\simeq$ IR divergence)
- A new observable, supersymmetric Rényi entropy

Future directions

- Can we define an entropic C-function which coincides with the Zamolodchikov's c-function in 2d?
- Is there a modified REE that is stationary at conformal fixed points?
- Proof of the a-theorem by using SSA of EE?