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The Seiberg-Witten map for a linear time-dependent background*

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Abstract: In this talk the Seiberg-Witten map for a time-dependent background related to a null-brane orbifold is studied. The commutation relations of the coordinates are linear, i.e. it is an example of the Lie algebra type. The equivalence map between the Kontsevich star product for this background and the Weyl-Moyal star product for a background with constant noncommutativity parameter is also studied. The method used to solve the Seiberg-Witten equations is cohomological, and is based on the determination of a coboundary operator and then of the corresponding homotopy operator.

*hep-th/0304030

based also on earlier works

hep-th/0105192, hep-th/0107225, hep-th/0206231

with B. Zumino, D. Brace, A. Pasqua, U. Varadarajan

Plan

- Noncommutative space-time, Weyl quantization procedure and star product
- Null-brane orbifold and Sethi-Hashimoto algebra
- Coordinate transformation and equivalence map
- Gauge theory on noncommutative spaces and Seiberg-Witten map
- Coboundary operator and corresponding homotopy operator
- Ambiguities of the SW map
- Solutions of the SW map for constant θ^{ij}
- SW Differential equation
- Deformation of the BRST operator
- Solutions of the SW map for linear θ^{ij}
- Conclusions and outlook

Noncommutative spaces

$$\begin{array}{ccc}
\text{Manifold } \mathcal{M} & \xrightarrow[\text{Theorem}]{\text{Gelfand Naimark}} & \begin{array}{c}\text{Commutative Algebra}\\ \text{of continous functions}\\ \mathcal{C}(\mathcal{M})\end{array} \\
\downarrow & \text{Deformation} & \downarrow \\
\begin{array}{c}\text{Noncommutative}\\ \text{``manifold''}\end{array} & \stackrel{?}{\longrightarrow} & \begin{array}{c}\text{Noncommutative}\\ \text{Algebra } \mathcal{A}\end{array}
\end{array}$$

Space-time commutation relations

$\mathcal{A}$ associative algebra generated by the coordinates $\{x^i\}$, $1 \leq i \leq D$,
D=space-time dimension,
with relations

$$\left[x^i, x^j\right] = i\ \theta^{ij}$$

where θ is an antisymmetric tensor

$$\theta^{ij} = -\theta^{ji}$$

satisfying the Jacobi identity

$$\partial^i \theta^{kl} + \partial^k \theta^{li} + \partial^l \theta^{ik} = 0$$

Weyl quantization procedure

Provides a representation of $\mathcal{A}$ in $\mathcal{C}(\mathcal{M})$

$$f(x^i) \in \mathcal{C}(\mathcal{M}) \to W(f)(\widehat{x}^i) \in \mathcal{A}$$

Starting from the Fourier transform $\tilde{f}(k)$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{D/2}} \int dx e^{-ik_j x^j} f(x)$$

the Weyl operator $W(f)$ is defined by

$$W(f) = \frac{1}{(2\pi)^{D/2}} \int dk e^{ik_i \widehat{x}^i} \tilde{f}(k)$$

with the substitution $x^i \in \mathcal{C}(\mathcal{M}) \to \widehat{x}^i \in \mathcal{A}$.

The star product $f \star g$

$$f \star g = \frac{1}{(2\pi)^D} \int dk\, dp\, e^{i(k_j + p_j + g_j(k,p))x^j} \tilde{f}(k)\, \tilde{g}(p)$$

is defined as the function corresponding to

$$W(f)W(g) = \frac{1}{(2\pi)^D} \int dk\, dp e^{ik_i \widehat{x}^i} e^{ip_j \widehat{x}^j} \tilde{f}(k)\, \tilde{g}(p)$$

with $g_j(k,p)$ the expression obtained through the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{(A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A-B,[A,B]]+\ldots)}$$

applied to $A = k_i \widehat{x}^i, B = p_j \widehat{x}^j$.

Main properties of the Weyl quantization:

- It automatically selects the most symmetric ordering of the elements $\hat{x}^i \in \mathcal{A}$.

- It naturally reproduces the Weyl-Moyal star product for the case of constant θ

$$
\begin{aligned}
f \star g &= f e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} g \\
&= fg + \frac{1}{2} i\theta^{ij}\partial_i f \partial_j g \\
&\quad -\frac{1}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k f\ \partial_j\partial_l g + O[\theta^3]
\end{aligned}
$$

with $\partial_i \equiv \frac{\partial}{\partial x^i}$.

- It coincides with Kontsevich quantization procedure for nilpotent Lie algebras and it is equivalent to it for the more general linear case. The difference with the Kontsevich star product in the linear case can be expressed in terms of "loop graphs".
(Kathotia, math.QA/9811174)

Relation between θ and the B-field in string theory

In string theory, a constant Poisson tensor θ^{ij} is related to the antisymmetric tensor B^{ij} in the Neveu-Schwarz sector by the formula

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B}\right)^{[ij]}$$

where [ ] antisymmetric part, g metric, α' string tension

In the limit of a large B-field $\alpha' B >> g$ then there is the simple relation

$$\theta^{ij} = \frac{1}{B_{ij}}$$

The null-brane orbifold

Liu, Moore, Seiberg, hep-th/0204168, hep-th/0206182

Generators: $\{x^i\}, i = 1, \ldots, 4$,
$x^1 = x^+$, $x^2 = x^-$, $x^3 = x$, $x^4 = z$
Orbifold identifications

$$z \sim z + \frac{2\pi}{\widetilde{R}},\ x^+ \sim x^+,\ x \sim x + 2\pi x^+$$
$$x^- \sim x^- + 2\pi x + \tfrac{1}{2}(2\pi)^2 x^+$$

The associated noncommutative algebra

Sethi, Hashimoto, hep-th/0208126

T-dualities and Twist
$\Rightarrow$ noncommutative algebra $\mathcal{A}$

Relations: $[x^i, x^j] = i\,\theta^{ij}$,
Noncommutativity parameter

$$\theta^{xz} = -\theta^{zx} = \widetilde{R}x^+,\ \ \theta^{x^-z} = -\theta^{zx^-} = \widetilde{R}x,$$
$$\theta^{ij} = 0 \text{ otherwise},\quad \widetilde{R} = \text{const},$$

Main properties of $\mathcal{A}$

- It is linear in the coordinates, i.e. it is of the Lie algebra type.

- θ^{ij} satisfies the Jacobi identity.

- It is a nilpotent algebra, i.e. the third commutator of any four elements vanishes.

$$[x^i,[x^j,[x^k,x^l]]] = 0 \quad \forall x^i \in \mathcal{A}$$

- It is noncommutative only in the xz and $x^- z$ directions.

The Kontsevich star product for $\mathcal{A}$

The Weyl quantization procedure gives

$$f \star g = f e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j} - \frac{1}{12}\theta^{ij}\partial_j\theta^{kl}\left(\overleftarrow{\partial_i}\overleftarrow{\partial_k}\overrightarrow{\partial_l} - \overleftarrow{\partial_k}\overrightarrow{\partial_i}\overrightarrow{\partial_l}\right)} g$$

Properties

- To the second order in θ it becomes

$$\begin{aligned} f \star g &= f\, g + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g - \frac{1}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k f \partial_j \partial_l g \\ &\quad - \frac{1}{12}\theta^{ij}\partial_j\theta^{kl}\left(\partial_i\partial_k f \partial_l g - \partial_k f \partial_i \partial_l g\right) + \ldots \end{aligned}$$

  which coincides with the expression given by Kontsevich (Kontsevich, q-alg/9709040).

- It is different than the Weyl-Moyal star product and generalizes it.

- It is an associative product.

The coordinate transformation

Liu, Moore, Seiberg, hep-th/0204168, hep-th/0206182

Through the coordinate transformation σ

$$x^+ = y^+;\; x = y^+\left(\tilde{y} + \widetilde{R}z\right);$$
$$x^- = y^- + \frac{1}{2}y^+\left(\tilde{y} + \widetilde{R}z\right)^2$$

the algebra $\mathcal{A}$ can be rewritten as the algebra generated by the elements $\{y^+, y, y^-, z\}$ with relations

$$[y^i, y^j] = i\,\tilde{\theta}^{ij}$$

where $y^1 = y^+, y^2 = y, y^3 = y^-, y^4 = z$ and

$$\tilde{\theta}^{\tilde{y}z} = -\tilde{\theta}^{z\tilde{y}} = \widetilde{R},\;\; \tilde{\theta}^{ij} = 0 \text{ otherwise},$$

and the orbifold identification becomes

$$y^+ \sim y^+;\; \tilde{y} \sim \tilde{y} + 2\pi;\; y^- \sim y^-;\; z \sim z + \frac{2\pi}{\widetilde{R}}.$$

This coordinate transformation is not linear and it is singular for $x^+ = y^+ = 0$. However, the orbifold identification is simpler and the noncommutativity parameter is constant.

The equivalence map

Two star products $\star$ and $\star'$ are equivalent if there exists an equivalence map, i.e. a differential operator R, such that

$$f \star' g = R^{-1}\left(R(f) \star R(g)\right)$$

Kontsevich's formality theorem:
two star products related by a coordinate transformation are equivalent in the above sense.
$\Rightarrow$ $\star$ and $\star' = \sigma(\tilde{\star})$ equivalent for $x^+ \neq 0$.
Expand $\star$, R in powers of θ

$$\begin{aligned} f \star g &= fg + B^{(1)}(f,g) + B^{(2)}(f,g) + \ldots \\ R(f) &= f + R^{(1)}(f) + R^{(2)}(f) + \ldots \\ \Rightarrow B'^{(1)}(f,g) &= B^{(1)}(f,g) + R^{(1)}(f)\, g \\ &\quad + f\, R^{(1)}(g) - R^{(1)}(f\, g) \end{aligned}$$

To the fourth order in θ, R is found to be

$$R^{(1)}(f) = R^{(3)}(f) = 0,$$

$$R^{(2)}(f) = \tfrac{1}{24}\theta^{zx}\partial_x\theta^{x^-z}\partial_{x^-},\; R^{(4)}(f) = {R^{(2)}}^2(f)$$

Main properties of the equivalence map

- To the fourth order R is generated by the flow of $R^{(2)}$, i.e. it is of the form:

$$R\,f = e^{-\frac{1}{24}\widetilde{R}^2 x^+ \partial_{x^-} \partial_z^2}\, f$$

- As expected, R is singular for $y^+ = x^+ = 0$, where the coordinate transformation σ is singular.

- The star product $\star$ is equivalent up to σ to the Weyl-Moyal product $\tilde{\star}$ for a constant $\tilde{\theta}^{ij}$
  $\Rightarrow$ $\star$ is associative.

- In principle R allows to map the results for a constant $\tilde{\theta}^{ij}$ to the algebra $\mathcal{A}$, e.g. the differential calculus, the SW map, instanton computations, at least outside of the singularity.

Gauge theory on noncommutative spaces and covariant coordinates

Wess et al., hep-th/0104153, hep-th/0001203,
Seiberg, hep-th/0008013

On commutative spaces

ψ gauge field, λ gauge parameter, a_i gauge potential

$$\delta a_i = \partial_i \lambda - i[a_i, \lambda]$$
$$\delta\psi = i\ \lambda\ \psi, \quad \delta x^i = 0 \Rightarrow \delta(x^i \psi) = i\lambda x^i \psi$$

On noncommutative spaces

Ψ gauge field, Λ gauge parameter, A^i gauge potential

$$\delta\Psi = i\,\Lambda \star \Psi$$
$$\Rightarrow\ \ \delta(x^i \star \Psi) = i\,x^i \star \Lambda \star \Psi \neq i\,\Lambda \star x^i \star \Psi$$

$\Rightarrow$ Introduce covariant coordinates X^i

$$X^i = x^i + A^i$$

such that

$$\delta X^i = i[\Lambda \stackrel{\star}{,} X^i]$$
$$\Rightarrow \delta(X^i \star \Psi) = i \star \Lambda \star X^i \star \Psi$$

This requires the noncommutative gauge transformation

$$\delta A^i = i[\Lambda \stackrel{\star}{,} x^i] + i[\Lambda \stackrel{\star}{,} A^i]$$

The SW map

Seiberg and Witten, JHEP09(1999)032

It expresses the noncommutative gauge field and parameter in terms of the commutative ones:

$$A^i = A^i(\theta, a, \partial a, \partial^2 a, \cdots),$$
$$\Lambda = \Lambda(\theta, \lambda, \partial\lambda, \cdots, a, \partial a, \cdots)$$

The functional dependence is determined by the SW equation

$$\delta A^i = i[\Lambda \stackrel{\star}{,} x^i] + i[\Lambda \stackrel{\star}{,} A^i]$$

For constant θ^{ij}, it is possible to identify

$$[x^i \stackrel{\star}{,} f] = i\theta^{ij}\partial_j f \text{ for all } f(x^i)$$

and rewrite

$$\delta A^i = \theta^{ij}\partial_j \Lambda + i[\Lambda \stackrel{\star}{,} A^i]$$

The index of the derivative is raised with θ^{ij}. It is a non-trivial result that this is consistent also for a θ^{ij} of the Lie algebra type, like the example we are studying here.

Properties of the SW map

- Usually the algebra of the gauge fields does not close in the noncommutative case and the fields are elements of the envelopping algebra. The SW map allows us to express an infinite number of noncommutative fields in terms of a finite number of commutative ones.

- In string theory the existence of the SW map follows from the fact that two different regularization techniques (Pauli-Villars and point-splitting) lead either to a commutative or a noncommutative theory and therefore the two theories are supposed to be physically equivalent.

- An interaction which is complicated when expressed in terms of the commutative variables becomes a simple free theory in the noncommutative coordinates. The interaction is encoded in the noncommutative structure of the space, i.e. in the geometry.

- There are different types of ambiguities in the solutions of the Seiberg-Witten equation, as a consequence of field redefinitions and the dependence on the choice of the path in θ-space.
  (Asakawa and Kishimoto, hep-th/9909139)

- From the physical point of view, the x^i are regarded as classical solutions, nontrivial vacuum expectation values, for coordinates on a D-brane, and the potential A^i is seen as a fluctuation around this value
  (Seiberg, hep-th/0008013).

Introduction of the ghost fields and of the coboundary operator

Instead of the gauge parameter λ use an odd ghost field v and define the BRST operator δ

$$\delta v = i v^2, \quad \delta a_i = \partial_i v - i[a_i, v] \equiv D_i v$$

with the properties

$$\delta^2 = 0, \qquad [\delta, \partial_i] = 0$$
$$\delta(f_1 f_2) = (\delta f_1) f_2 + (-1)^{deg(f_1)} f_1 (\delta f_2)$$

Moreover, define the coboundary operator

$$\Delta = \begin{cases} \delta - i\{v, \cdot\} & \text{on odd quantities} \\ \delta - i[v, \cdot] & \text{on even quantities} \end{cases}$$

so that

$$\Delta v = -i v^2, \qquad \Delta a_i = \partial_i v$$
$$\Delta^2 = 0, \qquad [\Delta, D_i] = 0$$
$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + (-1)^{deg(f_1)} f_1 (\Delta f_2)$$

Seiberg-Witten equation

$$\delta \Psi = i \Lambda \star \Psi, \quad \delta \Lambda = i \Lambda \star \Lambda$$
$$\delta A^i = i[\Lambda \stackrel{\star}{,} x^i] - i[A^i \stackrel{\star}{,} \Lambda]$$

The equation for Λ follows from the nilpotency of δ and the associativity of the star product.

Expansion in θ^{ij}

Gauge parameter and gauge field

$$\begin{aligned} \Lambda &= \Lambda^{(0)} + \Lambda^{(1)} + \ldots, & \Lambda^{(0)} &= v \\ A^i &= A^{i(1)} + A^{i(2)} + \ldots, & A^{i(1)} &= \theta^{ij} a_j \end{aligned}$$

The index of the gauge potential is raised with $\theta^{ij} \Rightarrow A^i$ starts at first order already.

General structure of the Seiberg-Witten equation to order n :

$$\Delta\Lambda^{(n)} = M^{(n)}, \quad \Delta A^{i(n)} = U^{i(n)}$$

Introduce the useful notation $b_i \equiv \partial_i v$.
Then for the gauge parameter

$$\Delta\Lambda^{(0)} = iv^2, \quad \Delta\Lambda^{(1)} = -\tfrac{1}{2}\theta^{ij} b_i b_j$$

$$\Delta\Lambda^{(2)} = -\tfrac{i}{8}\theta^{ij}\theta^{kl}\partial_i b_k \partial_j b_l - \tfrac{1}{2}\theta^{ij}[b_i, \partial_j \Lambda^{(1)}]$$

$$+ i\Lambda^{(1)}\Lambda^{(1)} - \tfrac{1}{12}\theta^{ij}\partial_j\theta^{kl}\{iD_i b_k - [a_i, b_k], b_l\}$$

and for the gauge potential

$$\begin{aligned} \Delta A^{i(1)} &= \theta^{ij} b_j, \\ \Delta A^{i(2)} &= \theta^{ij} D_j \Lambda^{(1)} - \frac{1}{2}\theta^{kl}\{b_k, \partial_l(\theta^{ij} a_j)\} \end{aligned}$$

The part of the SW equations which does not depend on derivatives of θ^{ij} decouples from the part which does and they can be solved separately.

$$\begin{aligned} M^{(n)} &= M'(\theta)^{(n)} + M''(\theta, \partial\theta)^{(n)}, \\ U^{i(n)} &= U^{i'}(\theta)^{(n)} + U^{i''}(\theta, \partial\theta)^{(n)} \end{aligned}$$

Consistency conditions following from $\Delta^2 = 0$:

$$\Delta M^{(n)} = 0, \quad \Delta U^{i(n)} = 0$$

$M'^{(n)}$ and $M''^{(n)}$, as well as $U^{i'(n)}$ and $U^{i''(n)}$ have to satisfy the consistency condition separately. $M''^{(n)}$ and $U^{i''(n)}$ satisfy it due to the Jacobi identity.

Ambiguities in the solutions

If Λ and A_i are solutions so are

$$\begin{aligned}\tilde{\Lambda}^{(n)} &= \Lambda^{(n)} + \Delta S^{(n)}, \\ \tilde{A}_i^{(n)} &= A_i^{(n)} + D_i S^{(n)} + S_i'^{(n)}\end{aligned}$$

for arbitrary $S^{(n)}$, $S_i'^{(n)}$ of ghost number 0 and $\Delta S_i' = 0$ (Asakawa and Kishimoto, hep-th/9909139)
The ambiguity due to S is of a gauge type, the one due to S' is of a covariant type.
The Seiberg-Witten equations are invariant under the noncommutative finite gauge transformations (Stora)

$$\begin{aligned}\Lambda &\to G^{-1}\Lambda G + i\,G^{-1}\delta_v G \\ A_i &\to G^{-1}A_i\,G + i\,G^{-1}\partial_i\,G \\ \Psi &\to G^{-1}\,\Psi\end{aligned}$$

where all products are star products,
G is an arbitrary element of ghost number 0.
The gauge ambiguities at the infinitesimal level can be recovered by choosing

$$G = 1 - iS^{(n)}$$

To first order

$$S^{(1)} = -i\theta^{ij}[a_i, a_j]$$

The homotopy operator

The consistency condition for the SW map suggests an analogy with the cohomology of chiral anomalies (Zumino, Les Houches lecture).

It is not possible to invert Δ, because it is nilpotent, but it is possible to construct the homotopy operator K satisfying

$$\Delta K + K \Delta = 1$$

Then

$$\Delta K M + K \Delta M = \Delta K M = M$$

and therefore $\Lambda = KM$ is a solution.

Only b_i and its derivatives enter in the equations, and never v itself. K is defined only on b_i.

Construction of K proceeds in two steps.
Basic variables: $a_i,\ b_i$
First, define infinitesimal version L
Action of L:

$$La_i = 0, \quad Lb_i = a_i, \quad [L, D_i] = 0$$
$$L(f_1 f_2) = (Lf_1)f_2 + (-1)^{deg(f_1)} f_1 (Lf_2)$$

L is odd and nilpotent: $L^2 = 0$.
Introduce $d =$ total order(monomial in a, b)

Then the homotopy operator K is defined:

$$K = D^{-1} L$$

with D^{-1} linear operator, which on monomials multiplies by $\frac{1}{d}$
K is nilpotent: $K^2 = 0$.
K is odd and has negative ghost number -1.
Example:

$$\begin{aligned}
\Lambda^{(1)} &= K(-\frac{1}{2}\theta^{ij} b_i b_j) = -\frac{1}{2}\theta^{ij} D^{-1} L(b_i b_j) \\
&= -\frac{1}{2}\theta^{ij} D^{-1}(a_i b_j - b_i a_j) \\
&= \frac{1}{4}\theta^{ij} \left\{ b_i, a_j \right\}
\end{aligned}$$

The constraints

The variables a_i and b_i are not free, because from $b_i \equiv \partial_i v$ it follows

$$\partial_i b_j - \partial_j b_i = 0$$

which is equivalent to

$$\Delta F_{ij} = D_i b_j - D_j b_i + i[b_i, a_j] + i[a_i, b_j] = 0$$

Analogously, the covariant derivatives have to satisfy the constraint

$$\left[F_{ij}, \cdot\right] - i[D_i, D_j](\cdot) = 0$$

Solution: Symmetrization procedure
Separate the symmetric part of $D^k a$ or $D^k b$ and substitute the constraints recursively for the antisymmetric pieces. For example:

$$D_i a_j \to \frac{1}{2}(D_i a_j + D_j a_i + F_{ij} - i[a_i, a_j])$$

Then treat F and its derivatives as scalars.
There are no independent constraints of higher order.

Solutions to second order for constant θ^{ij}

By applying the homotopy operator to the symmetrized $M^{(2)}$:

$$\begin{aligned}\Lambda^{(2)} &= -\frac{1}{2}\theta^{ij}\{a_i, \frac{1}{3}D_j\Lambda^{(1)} + \frac{i}{4}[a_j, \Lambda^{(1)}]\} \\ &+\theta^{ij}\theta^{kl}(-\frac{i}{16}[D_i a_k, D_j b_l] \\ &+[[a_i, a_k], \frac{1}{24}D_j b_l + \frac{i}{32}[a_j, b_l]] \\ &+\frac{1}{24}[D_i a_k, [a_j, b_l]] \\ &+\frac{1}{8}(a_i(\frac{1}{3}D_j a_k - \frac{1}{3}D_k a_j + \frac{i}{2}[a_j, a_k])b_l \\ &-b_i(\frac{1}{3}D_j a_k - \frac{1}{3}D_k a_j + \frac{i}{2}[a_j, a_k])a_l \\ &+\{\frac{1}{6}(D_i a_k - D_k a_i) + \frac{i}{4}[a_i, a_k], \{a_l, b_j\}\})) \ .\end{aligned}$$

Comparison between the solutions

A known solution (Munich group) is

$$\begin{aligned}\tilde{\Lambda}^{(2)} &= \frac{1}{32}\theta^{ij}\theta^{kl}\big(-\{b_i,\{a_k,i[a_j,a_l]+4\partial_l a_j\}\}\\ &\quad -i\{a_j,\{a_l,[b_i,a_k]\}\}+2i[[a_j,a_l],[b_i,a_k]]\\ &\quad +2[[b_i,a_k]+i\partial_i b_k,\partial_j a_l]\big)\end{aligned}$$

As expected, the two solutions $\Lambda^{(2)}$ and $\tilde{\Lambda}^{(2)}$ differ by an ambiguity $\Delta S^{(2)}$ with

$$\begin{aligned}S^{(2)} &= K(\Lambda^{(2)}-\tilde{\Lambda}^{(2)})\\ &= \theta^{ij}\theta^{kl}[(\frac{1}{24}([a_j,[D_i a_k,a_l]]\\ &\quad +2(D_i a_k a_j a_l + a_l a_j D_i a_k)\\ &\quad +\frac{1}{16}[a_i a_k,\Delta F_{jl}]]\ .\end{aligned}$$

The same technique can be applied to compute the gauge potential $A_i^{(2)}$ and to higher orders in θ^{ij}. It can be done e.g. by computer.

Seiberg-Witten differential equation

For θ^{ij} constant, let $\theta^{ij} \to t\,\theta^{ij}$.
The star product depends on an evolution parameter t.
Define new operators at “time” t:

$$\Delta_t = \begin{cases} \delta - i\{\Lambda \overset{\star}{,} \cdot\} & \text{on odd quantities} \\ \delta - i[\Lambda \overset{\star}{,} \cdot] & \text{on even quantities} \end{cases}$$

Covariant derivative $D_{i,t}$

$$D_{i,t} = \partial_i - i[A_i \overset{\star}{,} \cdot]$$

They have the properties

$$\Delta_t A_i = \partial_i \Lambda, \quad \Delta_t^2 = 0, \quad [\Delta_t, D_{i,t}] = 0$$

$$\Delta_t(f_1 f_2) = (\Delta_t f_1) f_2 + (-1)^{deg(f_1)} f_1 (\Delta_t f_2)$$

Differentiate the Seiberg-Witten equations

$$\begin{aligned} \Delta_t\, \dot{\Lambda} &= -\theta^{kl} \partial_k \Lambda \star \partial_l \Lambda \\ \Delta_t\, \dot{A}_i &= \Delta_i\, \dot{\Lambda} + \frac{1}{2} \theta^{kl} \{\partial_k A_i \overset{\star}{,} \partial_l \Lambda\} \end{aligned}$$

where $\dot{f} = \frac{df}{dt}$

Then a solution are the evolution equations

$$\begin{aligned}\dot{\Lambda} &= \frac{1}{4}\theta^{ij}\left\{\partial_i\Lambda, A_j\right\} \\ \dot{A}_i &= -\frac{1}{4}\theta^{kl}\left\{A_k, \partial_l A_i + F_{li}\right\}\end{aligned}$$

By differentiating $\dot{\Lambda}$

$$\begin{aligned}\ddot{\Lambda} = \ & \frac{1}{16}\theta^{ij}\theta^{kl}\Big(\left\{\left\{\partial_i\partial_k\Lambda \stackrel{\star}{,} A_j\right\} + \left\{\partial_i\Lambda \stackrel{\star}{,} \partial_k A_j\right\} A_l\right\} \\ & -\left\{\partial_i\Lambda \stackrel{\star}{,} \left\{A_k \stackrel{\star}{,} \partial_l A_j + F_{lj}\right\}\right\} \\ & +2i\left[\partial_i\partial_k\Lambda \stackrel{\star}{,} \partial_j A_l\right] \Big)\end{aligned}$$

We can compute $\dfrac{d^n\Lambda}{dt^n}$. We can obtain solutions $\Lambda^{(n)}$ as

$$\Lambda^{(n)} = \frac{1}{n!}\frac{d^n\Lambda}{dt^n}$$

The solution to second order obtained from $\ddot{\Lambda}$ again differs from $\Lambda^{(2)}$ by an ambiguity.

With this method the homotopy operator has to be applied at most at first order: no problems with constraints.

Deformation of the BRST operator

(Weinstein)

Start with the original BRST operator

$$\delta_0 \Lambda = i\Lambda \star \Lambda$$

and deform it to

$$\delta = \delta_0 + \delta_1 + \delta_2 + \ldots$$

such that

$$\delta v = iv \star v, \quad \delta^2 = 0$$

Introduce the deforming map D such that

$$Dv = \Lambda$$

and

$$D\delta = \delta_0 D, \quad [D, \partial] = 0$$

with the properties

$$D(fg) = D(f)D(g), \quad D(f \star g) = D(f) \star D(g)$$

Expand in θ^{ij}

$$D = 1 + D^{(1)} + D^{(2)} + \dots$$

where $D^{(1)}$ is a vector field

$$D^{(1)}(fg) = D^{(1)}(f)g + fD^{(1)}(g)$$

Order by order in θ^{ij} the equation

$$D\delta = \delta_0 D$$

becomes

$$\begin{aligned} \delta_1 &= \delta_0 D^{(1)} - D^{(1)}\delta_0 \\ \delta_2 &= \delta_0 D^{(2)} - D^{(2)}\delta_0 - D^{(1)}\delta_1 \end{aligned}$$

with

$$D^{(n)}v = \Lambda^{(n)}, \quad \delta_n v = B^{(n)}(v,v)$$

and we recover the equation $\Delta\Lambda^{(n)} = M^{(n)}$.

The equation $D\delta = \delta_0 D$ is nothing else than the definition of an algebroid morphism.

Solution for the gauge parameter for linear θ^{ij}

To the first order there is no correction due to the fact that θ^{ij} is not constant and we recover the usual result

$$\Lambda^{(1)} = \frac{1}{4}\theta^{kl}\{b_k, a_l\}$$

To the second order we recover the previous result

$$\begin{aligned}\Lambda'^{(2)} = -\tfrac{1}{2}\theta^{ij}\{a_i, \tfrac{1}{3}D_j\Lambda^{(1)} + \tfrac{i}{4}[a_j, \Lambda^{(1)}]\}\\ +\theta^{ij}\theta^{kl}\big(-\tfrac{i}{16}[D_i a_k, D_j b_l] + [[a_i, a_k], \tfrac{1}{24}D_j b_l\\ +\tfrac{i}{32}[a_j, b_l]] + \tfrac{1}{24}[D_i a_k, [a_j, b_l]]\\ +\tfrac{1}{8}(a_i(\tfrac{1}{3}D_j a_k - \tfrac{1}{3}D_k a_j + \tfrac{i}{2}[a_j, a_k])b_l\\ -b_i(\tfrac{1}{3}D_j a_k - \tfrac{1}{3}D_k a_j + \tfrac{i}{2}[a_j, a_k])a_l\\ +\{\tfrac{1}{6}(D_i a_k - D_k a_i) + \tfrac{i}{4}[a_i, a_k], \{a_l, b_j\}\})\big)\end{aligned}$$

and we find the correction

$$\begin{aligned}\Lambda''^{(2)} = -\tfrac{1}{4}\theta^{ij}\partial_j\theta^{kl}\Big(\tfrac{1}{6}(\{a_i, \{b_k, a_l\}\} + i[D_i a_k, b_l]\\ -i[D_i b_k, a_l]) + \tfrac{1}{9}\left([[a_i, b_k], a_l] - [[a_i, a_k], b_l]\right)\Big)\end{aligned}$$

Solution for the gauge potential

$$A'^{i(2)} = -\frac{1}{4}\theta^{ij}\theta^{kl}\{a_k, \partial_l a_j + F_{lj}\}$$
$$A''^{i(2)} = -\frac{1}{4}\theta^{kl}\partial_l\theta^{ij}\{a_k, a_j\}$$

For the algebra $\mathcal{A}$ of Sethi and Hashimoto

$$A''^{i(2)} = \begin{cases} \frac{1}{4}\widetilde{R}^2 x^+\{a_z, a_z\} & \text{for } i = x^-, \\ -\frac{1}{4}\widetilde{R}^2 x^+\{a_z, a_{x^-}\} & \text{for } i = z, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of an abelian gauge theory

$$\begin{aligned} A''^{i(3)} = \frac{1}{6}\Big(&-\theta^{ij}\partial_j\theta^{rs}\theta^{kl}a_k a_r(\partial_l a_s + \partial_s a_l) \\ &+\theta^{sj}\partial_j\theta^{ir}\theta^{kl}a_k a_r\,(2\partial_l a_s - \partial_s a_l) \\ &+\theta^{rj}\partial_j\theta^{si}\theta^{kl}a_r a_k\,(2\partial_s a_l - 4\partial_l a_s)\Big) \\ &+\frac{1}{6}\theta^{ij}\theta^{kl}\partial_l\theta^{rs}a_k a_s\left(2\partial_j a_r - 3\partial_r a_j\right) \\ &+\frac{1}{6}\theta^{kl}\partial_l\theta^{rs}\partial_s\theta^{ij}a_k a_r a_j \end{aligned}$$

Conclusions and outlook

- With a cohomological approach the solutions to the SW equations can be computed for each gauge group and to each order in θ^{ij}.

- In some instances, as in the case of the null-brane orbifold studied here, it can also be used for a time-dependent background or a more general non-constant θ^{ij}.

- By using ghosts a connection with the Batalin-Vilkoviskij formalism can be made. The SW map could be formulated in terms of a master equation (see Barnich, Grigoriev, Henneaux, hep-th/0106188)

- This type of cohomology is related to an algebroid structure, because it comes from the action of Lie algebra on the fields. In other words, it is possible to deform the BRST operator δ itself rather than the gauge parameter and the gauge potential. It is an equivalent approach (Weinstein).

- It is also related to supermanifolds, so that it could be used to formulate and study the properties of the SW map for supersymmetric theories.

- The Weyl-Moyal product appears in string field theory, because it is related to the Witten star product. Noncommutative geometry may prove relevant in this context (see e.g. Bars, Deliduman, Pasqua, Zumino, hep-th/0308107).

- Through the use of cohomology the renormalization properties of a noncommutative gauge theory could be studied. Expanding in θ is a way to get around the infrared-ultraviolet mixing occurring in noncommutative field theories (see Grosse with Vienna group, hep-th/0104097)