# A SCALING LIMIT WITH MANY NONCOMMUTATIVITY PARAMETERS

1. $U(N) \to U(1)^N$ MAGNETIC BACKGROUND
2. CHARGED STRING PROPAGATOR
3. SET OF NONCOMMUTATIVITY PARAMETERS
4. STAR PRODUCTS FROM OPE'S
5. SCALING LIMIT

L.D., C. NAPPI hep-th/0009225

# OPEN STRINGS

NEUTRAL STRING
IN BACKGROUND B-FIELD

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu \quad - \frac{i}{2} \int_\Sigma B_{ij} \varepsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j$$

$$= \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu \quad - \frac{i}{2} \int_{\partial\Sigma} B_{ij} X^i \partial_t X^j$$

HAS SCALING LIMIT = N.C. YANG-MILLS

CHARGED STRING
(WITH A DIFFERENT MAGNETIC FIELD AT EACH END)

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu$$
$$- \frac{i}{2} \int_{-\infty}^{\infty} dt \left( B^{(1)}_{ij} X^i \partial_t X^j \Big|_{\sigma=0} + B^{(2)}_{ij} X^i \partial_t X^j \Big|_{\sigma=\pi} \right)$$

$0 \le \mu \le 25$, $1 \le i \le p$    Dp-branes in $(0, i)$ directions

SCALING LIMIT = N.C. $U(1)^N$
WITH MASSIVE CHARGED STATES

## EQ'S OF MOTION:

$B^{(\ell)}_{ij}$ CONSTANT,

EUCLIDEAN WORLSHEET METRIC

$$(\partial_\sigma^2 + \partial_t^2)\, X^\mu(\sigma, t) = 0$$

## BOUNDARY CONDITIONS:

$$g_{ij}\partial_\sigma X^j + 2\pi i\alpha' B^{(1)}_{ij} X^j \Big|_{\sigma=0} = 0$$

$$g_{ij}\partial_\sigma X^j - 2\pi i\alpha' B^{(2)}_{ij} X^j \Big|_{\sigma=\pi} = 0$$

DIFFER AT EACH END OF THE STRING.

$$\partial_\sigma X^0\Big|_{\sigma=0,\pi} = 0 \quad ; \quad X^I\Big|_{\sigma=0,\pi} = 0, \quad p+1 \le I \le 25.$$

DIRECTIONS TRANSVERSE TO Dp-branes

NEUTRAL STRING RESTORED FOR $B^{(1)}_{ij} = -B^{(2)}_{ij}$

NOW LET $1 \le i, j \le 2$. <u>D2 - D2 system</u>

$B^{(1)}_{12} = q_1 B_{12}$, $B^{(2)}_{12} = q_2 B_{12}$ ; $g_{ij} = g^{-1}\delta_{ij}$ ; $G_{ij} = G^{-1}\delta_{ij}$

# CHARGED STRING NORMAL MODE EXPANSION

IN $X^{\pm}(\sigma,t) \equiv X^1 \pm i X^2(\sigma,\tau)$ BASIS :

$$X^+(z,\bar{z}) = x^+ + \frac{i}{2}\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+A} \frac{a_r}{r}\left(z^{-r}+\bar{z}^{-r}\right)$$
$$-\frac{1}{2}\sqrt{2\alpha'}\, B \sum_{r\in\mathbb{Z}+A} \frac{a_r}{r}\left(z^{-r}-\bar{z}^{-r}\right)$$

$$X^-(z,\bar{z}) = x^- + \frac{i}{2}\sqrt{2\alpha'} \sum_{s\in\mathbb{Z}-A} \frac{\hat{a}_s}{s}\left(z^{-s}+\bar{z}^{-s}\right)$$
$$+\frac{1}{2}\sqrt{2\alpha'}\, B \sum_{s\in\mathbb{Z}-A} \frac{\hat{a}_s}{s}\left(z^{-s}-\bar{z}^{-s}\right).$$

## CHARGED STRING COMMUTATION RELATIONS:

$$[a_r, \hat{a}_s] = 2G\, r\, \delta_{r,-s} \quad ; \quad [a_r, a_{r'}] = 0$$
$$[\hat{a}_s, \hat{a}_{s'}] = 0$$
$$[x^+, x^-] = -\frac{2}{(q_1+q_2)B_{12}} \qquad [a_r, x^{\pm}] = 0$$
$$[\hat{a}_s, x^{\pm}] = 0 .$$

(ACNY)

DEFINITIONS :

$$G = \frac{g}{1+B^2}$$

$$B \equiv g \, q_1 \, 2\pi\alpha' B_{12}$$

$$A = \frac{1}{\pi}\left( \arctan B + \arctan \frac{q_2}{q_1} B \right)$$

NON-INTEGER MODED COMPLEX W.S. BOSONS:

$a_r \quad : \quad r \in \mathbb{Z} + A$

$\tilde{a}_s \quad : \quad s \in \mathbb{Z} - A$

$$a_r^{+} = \tilde{a}_{-r}$$

$A \to 0 \quad \sim \quad q_1 \to -q_2$

LIMIT BACK TO NEUTRAL STRING :

$$\lim_{A\to 0} X^{\pm}(z,\bar{z}) = X^{\pm}_{n.s.}(z,\bar{z}) \qquad a_A \to a_0 \sim p_0$$

WHEN

$$\begin{cases} \lim\limits_{A\to 0}\left( x^{+} + i\sqrt{2\alpha'}\,\dfrac{a_A}{A} \right) \equiv x_0^{+} \\ \lim\limits_{A\to 0}\left( x^{-} - i\sqrt{2\alpha'}\,\dfrac{a_{-A}}{A} \right) \equiv x_0^{-} . \end{cases}$$

CHARGED STRING OPERATORS :

$$x^{\pm}, \; a_A, \; \tilde{a}_{-A} \; ; \; a_r, \tilde{a}_s$$

NEUTRAL STRING OPERATORS :

$$x_0^{\pm}, \; p_0^+ \equiv \frac{a_0}{\sqrt{2\alpha'}}, \; p_0^- \equiv \frac{\tilde{a}_0}{\sqrt{2\alpha'}} \; ; \; a_n, \tilde{a}_m$$

$n, m \in \mathbb{Z}$.

---

NEUTRAL STRING COMMUTATION RELATIONS :

$$[a_n, \tilde{a}_m] = 2G \, n \, \delta_{n,-m}$$

$$[x_0^+, x_0^-] = 2\Theta^{12}$$

$$[x_0^+, p_0^-] = i2G = [x_0^-, p_0^+]$$

$$[a_n, a_m] = 0$$
$$[\tilde{a}_n, \tilde{a}_m] = 0$$

FOR $n \neq 0$:

$$[a_n, x_0^{\pm}] = 0$$
$$[\tilde{a}_m, x_0^{\pm}] = 0$$

$$[a_0, x_0^+] = 0$$
$$[\tilde{a}_0, x_0^-] = 0$$

$$\Theta^{12} = -2\pi\alpha' B G = \frac{-2\pi\alpha'(g)^2 \, q_1 2\pi\alpha' B_{12}}{1 + (g q_1 2\pi\alpha' B_{12})^2}$$

# CHARGED STRING SPECTRUM

Let $a(z) \equiv \sum_{r\in\mathbb{Z}+A} a_r z^{-r-1}$

$\tilde{a}(z) \equiv \sum_{s\in\mathbb{Z}-A} \tilde{a}_s z^{-s-1}$

VIRASORO ALG.

$$L(z) = \tfrac{1}{2} G^{-1} :a(z)\tilde{a}(z): + z^{-2}\tfrac{1}{2}A(1-A)$$

(CENTRAL CHARGE = 2)

SINCE NO $\hat{P}^{\pm}$,
LET STATES BE
EIGENSTATES OF POSITION OPERATOR

$$|x_+\rangle \equiv e^{-\frac{1}{2}(q_1+q_2)B_{12}\, x_+ \hat{x}_-} |0\rangle$$

$\Rightarrow \hat{x}^+ |x_+\rangle = x_+ |x_+\rangle$

## NORMAL ORDERING

Let $a_r |\chi_+\rangle = 0 \qquad r \geq A$

$\tilde{a}_s |\chi_+\rangle = 0 \qquad s \geq 1-A$

$0 \leq A \leq \frac{1}{2}$.

$$L_0 |\chi_+\rangle = \tfrac{1}{2} A(1-A) |\chi_+\rangle$$

FOR EACH OF THE INFINITE NUMBER OF STATES $|\chi_+\rangle$.

## OSCILLATOR TOWER:

| STATE | $\alpha' M^2 = L_0 - 1$ |
|---|---|
| $\lvert\chi_+\rangle$ | $-1 + \frac{1}{2}A(1-A) < 0$ |
| $a_{-1+A} \lvert\chi_+\rangle$ | $-1 + \frac{1}{2}A(1-A) + (1-A) = -\frac{1}{2}A(1+A) < 0$ |
| $\tilde{a}_{-1-A} \lvert\chi_+\rangle$ | $-1 + \frac{1}{2}A(1-A) + (1+A) = \frac{1}{2}A(3-A) > 0$ |
| $\vdots$ | |

POLARIZATIONS ARE SPLIT IN FREQUENCY:

1 TACHYONIC MODE

1 MASSIVE MODE

# LANDAU LEVELS: $\quad (a_A |x_+\rangle = 0)$

$$|x_+\rangle, \quad \tilde{a}_{-A} |x_+\rangle, \quad \tilde{a}_{-A} \tilde{a}_{-A} |x_+\rangle, \ldots$$

$$\Delta(L_0 - 1) = A$$

SO THESE STATES HAVE FREQUENCY SEPARATION A.

THEY ARE LIKE EQUALLY SPACED LANDAU LEVELS (EACH) OF INFINITE DEGENERACY.

AND EACH IS THE LOWEST STATE OF AN OSCILLATOR TOWER.

TO COMPUTE
CHARGED STRING PROPAGATOR.

CONSIDER THE COHERENT STATES:

$$|\alpha\rangle = e^{-\frac{1}{2}(q_1+q_2)B_{12}\, x^-}\, |0\rangle$$

$$\langle\beta| = \langle 0|\, e^{-1/2\left(x^+ + i\frac{\sqrt{2\alpha'}}{A} a_A\right)}$$

THIS WILL ENSURE
CHARGED PROPAGATOR $\xrightarrow[A\to 0]{}$ NEUTRAL PROP.

SINCE $\lim_{A\to 0} \langle\beta|\, x_0^+ x_0^- + x_0^- x_0^+\, |\alpha\rangle = 0.$

$\&$ $N\, x_0^+ x_0 \equiv \frac{1}{2}(x_0^+ x_0^- + x_0^- x_0^+)$
IN NEUTRAL STRING.

## CHARGED STRING PROPAGATOR:

For $|z| > |\varsigma|$:

$$\langle X^+(z,\bar z)\, X^-(\varsigma,\bar\varsigma)\rangle \equiv \langle \beta |\, X^+(z,\bar z)\, X^-(\varsigma,\bar\varsigma)\,|\alpha\rangle$$

$$= -\frac{2\alpha'\pi g}{B+\frac{q_2}{q_1}B} - 2\alpha' G \frac{1}{A}\left(\varsigma^A+\bar\varsigma^A-1\right) + 2i\alpha' G B \frac{1}{A}\left(\varsigma^A-\bar\varsigma^A\right)$$

$$+ \alpha' G \left[ f\left(\frac{\varsigma}{z}\right) + f\left(\frac{\bar\varsigma}{\bar z}\right) + f\left(\frac{\varsigma}{\bar z}\right) + f\left(\frac{\bar\varsigma}{z}\right) \right]$$

$$+ \alpha' G B^2 \left[ f\left(\frac{\varsigma}{z}\right) + f\left(\frac{\bar\varsigma}{\bar z}\right) - f\left(\frac{\varsigma}{\bar z}\right) - f\left(\frac{\bar\varsigma}{z}\right) \right]$$

$$+ 2i\alpha' G B \left[ -f\left(\frac{\varsigma}{z}\right) + f\left(\frac{\bar\varsigma}{\bar z}\right) \right]$$

where

$$f(\rho) \equiv \sum_{\substack{r=n+A\\ n\geq 0}} \frac{\rho^r}{r} \quad ; \quad \lim_{A\to 0} f(\rho) = -\ln(1-\rho) + \lim_{A\to 0}\frac{\rho^A}{A}$$

(INCOMPLETE BETA FUNCTION)

OTHER NON ZERO COMPONENT: $|z|>|s|$

$$\langle X^-(z,\bar z)\, X^+(s,\bar s)\rangle \equiv \langle B|\, X^-(z,\bar z)\, X^+(s,\bar s)\,|B\rangle$$

$$= \frac{2\alpha'\pi g}{B+\frac{g_2}{g_1}B} - 2\alpha' G\,\frac{1}{A}\left(z^A+\bar z^A-1\right) + 2i\alpha' G\,\frac{B}{A}\left(z^A-\bar z^A\right)$$

$$+\alpha' G\left[\, g\left(\frac{s}{z}\right)+g\left(\frac{\bar s}{\bar z}\right)+g\left(\frac{s}{\bar z}\right)+g\left(\frac{\bar s}{z}\right)\right]$$

$$+\alpha' G B^2\left[\, g\left(\frac{s}{z}\right)+g\left(\frac{\bar s}{\bar z}\right)-g\left(\frac{s}{\bar z}\right)-g\left(\frac{\bar s}{z}\right)\right]$$

$$-2i\alpha' G B\left[\, -g\left(\frac{s}{\bar z}\right)+g\left(\frac{\bar s}{z}\right)\right].$$

$$g(\rho) \equiv \sum_{\substack{s=n-A\\ n>0}} \frac{\rho^s}{s} \quad ; \quad \lim_{A\to 0} g(\rho) = -\ln(1-\rho).$$

COMPARISON WITH NEUTRAL STRING PROPAGATOR:

$$\langle x^i(z,\bar z)\, x^j(\varsigma,\bar\varsigma)\rangle_{N.S.}$$

$$= -\alpha' \Big[ \tfrac{1}{2} g^{ij} \ln(z-\varsigma) + \tfrac{1}{2} g^{ij} \ln(\bar z - \bar\varsigma)$$
$$+\Big(-\tfrac{1}{2} g^{ij} + G^{ij} + \frac{\Theta^{ij}}{2\pi\alpha'}\Big) \ln(z-\bar\varsigma)$$
$$+\Big(-\tfrac{1}{2} g^{ij} + G^{ij} - \frac{\Theta^{ij}}{2\pi\alpha'}\Big) \ln(\bar z-\varsigma)$$
$$-\frac{i}{2\alpha'}\Theta^{ij} \Big].$$

CHARGED PROPAGATOR FOR ALL $z, \varsigma$:

$$G^{+-}(z,\bar z;\varsigma,\bar\varsigma) = \Theta(|z|-|\varsigma|)\langle x^+(z,\bar z)\, x^-(\varsigma,\bar\varsigma)\rangle$$
$$+\Theta(|\varsigma|-|z|)\langle x^-(\varsigma,\bar\varsigma)\, x^+(z,\bar z)\rangle$$

AND $G^{-+}(z,\bar z;\varsigma,\bar\varsigma) = G^{+-}(\varsigma,\bar\varsigma; z,\bar z)$.

$$\Rightarrow \quad \lim_{A\to 0} G^{+-} = \langle x^+ x^-\rangle_{N.S.}$$

$$\lim_{A\to 0} G^{-+} = \langle x^- x^+\rangle_{N.S.}\,.$$

# COMPUTATION OF NONCOMMUTATIVITY PARAMETERS:

1. CALCULATE THE EQUAL TIME COMMUTATOR VIA A SHORT DISTANCE EXPANSION OF THE PROPAGATOR :

2. EVALUATE PROPAGATOR ON EACH BOUNDARY

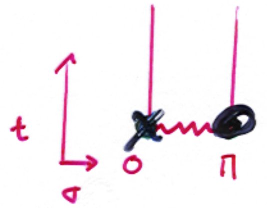
$$\Rightarrow 2\Theta^{12} \equiv [x^+(\tau), x^-(\tau)]$$

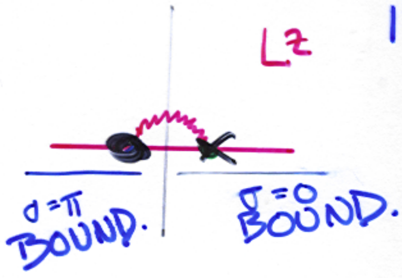
$$= T\left(x^+(\tau)x^-(\tau^-) - x^+(\tau)x^-(\tau^+)\right)$$

$$= \lim_{\epsilon \to 0} \langle x^+(\tau)x^-(\tau-\epsilon)\rangle - \langle x^-(\tau+\epsilon)x^+(\tau)\rangle$$

$\uparrow$

PROPAGATORS




FOR EXAMPLE :

ON $\sigma = 0$ BOUNDARY

$\Rightarrow \quad z = |z| = \tau > 0 \quad , \quad \zeta = |\zeta| = \tau' > 0$

ON THE BOUNDARY $\sigma=0$
<u>THE PROPAGATOR IS</u> :

$$\langle X^+(z,\bar{z})\, X^-(\zeta,\bar{\zeta})\rangle\Big|_{\sigma=0} = \frac{-2\alpha'\pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \sum_{n=0}^{\infty} \frac{1}{n+A}\left(\frac{\zeta}{z}\right)^{n+A} + \frac{2\alpha' G}{A} - \frac{4\alpha' G}{A}\zeta^A$$

$$\langle X^-(z,\bar{z})\, X^+(\zeta,\bar{\zeta})\rangle\Big|_{\sigma=0} = \frac{2\alpha'\pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \sum_{n=1}^{\infty} \frac{1}{n-A}\left(\frac{\zeta}{z}\right)^{n-A} + \frac{2\alpha' G}{A} - \frac{4\alpha' G}{A} z^A .$$

$$[X^+(\tau), X^-(\tau)]\Big|_{\sigma=0}$$

$$= \lim_{\epsilon \to 0} \left( \frac{-4\alpha'\pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \left[ \sum_{n=0}^{\infty} \frac{1}{n+A} \left( \frac{\tau-\epsilon}{\tau} \right)^{n+A} - \sum_{n=1}^{\infty} \frac{1}{n-A} \left( \frac{\tau}{\tau+\epsilon} \right)^{n-A} \right] \right)$$

$$= \frac{-4\alpha'\pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \pi \cot \pi A$$

$$= \boxed{-4\alpha'\pi (g)^2 \frac{q_1 2\pi\alpha' B_{12}}{1 + (g q_1 2\pi\alpha' B_{12})^2}}$$

$$\equiv 2\Theta^{12}$$

THE NONCOMMUTATIVITY PARAMETER AT $\sigma = 0$

AT THE $\sigma = \pi$ BOUNDARY,

$$z = |z| e^{i\pi} = \tau < 0 \quad , \quad \zeta = |\zeta| e^{i\pi} = \tau' < 0$$

$$\Rightarrow \quad [X^+(\tau), X^-(\tau)]\Big|_{\sigma=\pi}$$

$$= \lim_{\epsilon \to 0} \left( \langle X^+(\tau) X^-(\tau+\epsilon) \rangle - \langle X^-(\tau-\epsilon) X^+(\tau) \rangle \right)\Big|_{\sigma=\pi}$$

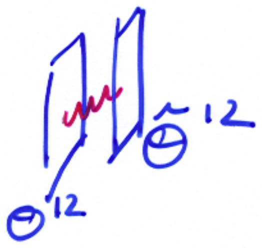
$$= \frac{-4\alpha' \pi (g)^2 \; q_2 \pi \alpha' B_{12}}{1 + (g)^2 (q_2 2\pi\alpha' B_{12})^2}$$

$$\equiv 2\tilde{\Theta}^{12}$$

THE NONCOMMUTATIVITY PARAMETER AT $\sigma = \pi$ BOUNDARY.

---

FOR COINCIDENT D2 branes,

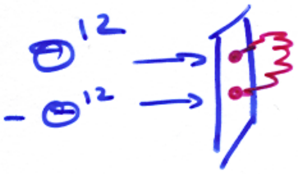



EACH BRANE HAS ITS OWN NONCOMMUTATIVITY PARAMETER.

FOR NEUTRAL STRING LIMIT:

$$q_1 = -q_2$$

$$\Rightarrow \tilde{\Theta}^{12} = -\Theta^{12}$$




BOTH ENDS OF THE STRING ARE ON THE SAME D2-BRANE.

THE NC PARAMETER AT ONE END OF THE STRING IS EQUAL TO MINUS THAT AT THE OTHER END.

# SHORT DISTANCE BEHAVIOR AND STAR PRODUCTS :

OPE FOR TACHYON VERTEX OP.'S :
(ON $\sigma = 0$ BOUNDARY)

$$e^{ip\cdot X(\tau)} \, e^{iq\cdot X(\tau')}$$

$$= (\tau-\tau')^{4\alpha' G\,(p_- q_+ + p_+ q_-)}$$

$$\cdot\, e^{-\Theta^{12}(p_- q_+ - p_+ q_-)} \; :e^{i(p+q)X(\tau')}:$$

IN SCALING LIMIT $(\alpha' \to 0)$ ( $G, \Theta^{12}$ FIXED)

$$e^{ip\cdot X(\tau)} \, e^{iq\cdot X(\tau')} \sim e^{ip\cdot X(\tau')} * e^{iq\cdot X(\tau')}$$

STAR PRODUCT :

$$f(x) * g(x) = e^{\frac{i}{2}\Theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)\, g(x+\zeta)\Big|_{\xi=\zeta=0}$$

$$= fg + \frac{i}{2}\Theta^{ij}\,\partial_i f\,\partial_j g + O(\Theta^2)$$

$\Rightarrow$ A DIFFERENT STAR PRODUCT FOR EACH BOUNDARY.

THE SPECTRUM IN THE SCALING LIMIT:

$$g^{-1} \to \epsilon \ , \quad \alpha' \to \sqrt{\epsilon} \qquad \text{AS } \epsilon \to 0$$

$(\alpha' B_{12} \to \sqrt{\epsilon}$ ; keeping $B_{12}$ fixed).

$$\Rightarrow \quad \Theta^{12} \to \frac{1}{q_1 B_{12}} \ , \quad \hat{\Theta}^{12} \to \frac{1}{q_2 B_{12}} \ , \quad G \to \frac{1}{(q_1 2\pi)^2} .$$

THESE PARAMETERS ARE FINITE IN THE SCALING LIMIT.

$$A \to -\frac{(q_1+q_2)\sqrt{\epsilon}}{2\pi^2 q_1 q_2}$$

MASSES OF CHARGED VECTOR STATES :

$$a_{-1+A} \, |\chi_+\rangle \qquad m^2 = -\frac{1}{2\alpha'} A(1+A)$$

$$\to \frac{1}{2} \frac{(q_1+q_2) B_{12}}{2\pi^2 q_1 q_2} = \frac{(q_1+q_2) G}{q_2 \Theta^{12}}$$

$$\hat{a}_{-1-A} \, |\chi_+\rangle \qquad m^2 = \frac{1}{2\alpha'} A(3-A)$$

$$\to -\frac{3}{2} \frac{(q_1+q_2) B_{12}}{2\pi^2 q_1 q_2} = -3\frac{(q_1+q_2) G}{q_2 \Theta^{12}} .$$

CHARGED VECTOR STATES THAT SURVIVE IN THE SCALING LIMIT:

$$a_{-1+A}\,|\chi_+\rangle\,,\quad \hat{a}_{-1-A}\,|\chi_+\rangle$$

$$a_{-1+A}\,\hat{a}_{-A}\,|\chi_+\rangle\,,\quad \hat{a}_{-1-A}\,\hat{a}_{-A}\,|\chi_+\rangle$$

$\vdots$ VECTORS AT ALL LANDAU LEVELS.

COMPLETE SPECTRUM OF THE $U(1)^N$ NON COMMUTATIVE GAUGE THEORY

( $U(N)$ GAUGE THEORY EXPANDED AROUND A $U(1)^N$ BACKGROUND ):

$N$ NEUTRAL MASSLESS GLUONS

$N^2-N$ CHARGED VECTOR SECTORS