# Introduction to Loop Quantum Gravity

University of North Carolina, 3/4/2004

Hanno Sahlmann, Penn State University

## Introduction

Loop quantum gravity (Ashtekar, Rovelli, Smolin, ...):

- background independent
- minimalist

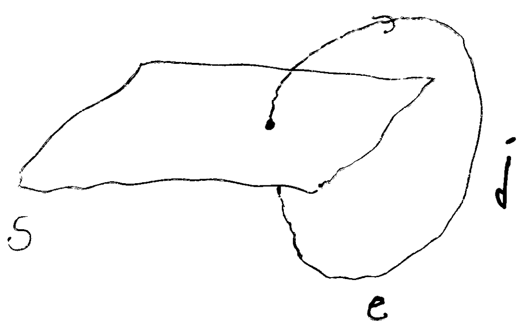
Two main ingredients,

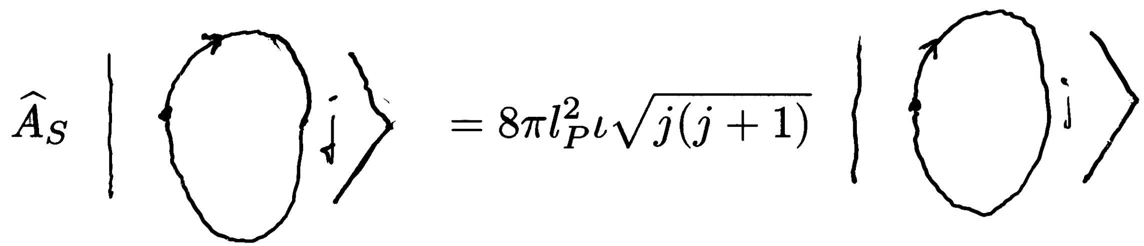
| gauge theoretic framework for GR | + | clever canonical quantization prescription |
|---|---|---|

lead to remarkably combinatorial picture of geometry:

States: Graphs with "colored" edges.
Operators $\widehat{A}$, $\widehat{V}$,... for area, volume,...

Example:



## Classical Theory

Work in Hamilton formulation, so: 3+1 split. $M = \mathbb{R} \times \Sigma$.
Canonical pair (*Ashtekar variables*): ($i, j, \ldots$: su(2)-indices, $a, b, \ldots$ space indices)

$A_a^i(p)$: SU(2) connection on $\Sigma$,

$E_i^a(p)$: (densitized) frame field on $\Sigma$.

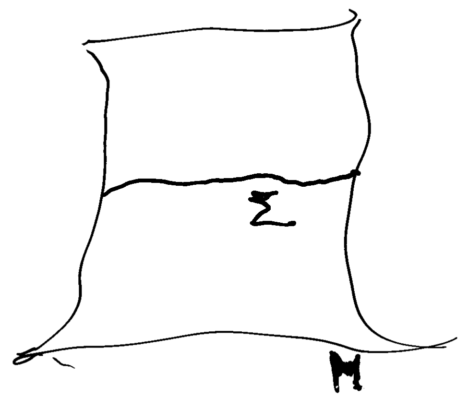
$$\{E_i^a(p), A_b^j(q)\} = \delta_b^a \delta_j^i \delta(p,q) \cdot 8\pi G$$

Relation to ADM variables $K_{ab}$, $q^{ab}$:

$$\det(q) q^{ab} = \iota E_i^a E^{bi}, \qquad A_a^i = \Gamma_a^i - \frac{\iota}{\sqrt{\det(q)}} K_{ab} E^{bi}$$

To recover the physical phase space, have to impose

1. Gauss constraint,
2. diffeomorphism constraint,
3. Hamiltonian constraint.



**Smearing:**

Can integrate $A$ naturally (i.e. without background structure) along path:

$$\left.\begin{array}{l}\text{Connection } A \text{ on } \Sigma,\\ \text{Path } e \text{ in } \Sigma\end{array}\right\} \longrightarrow \quad h_e[A] := \mathcal{P}\exp i\int_e A \qquad \text{“holonomy along } e\text{”}$$

$\tilde{E}_{abi} = E_i^c \epsilon_{cab}$ is 2-form. Can integrate naturally (i.e. without background structure) over surface:

$$\left.\begin{array}{l}\text{Triad } E \text{ on } \Sigma,\\ \text{Surface } S \text{ embedded in } \Sigma,\\ \text{su(2) valued function } f \text{ on } S\end{array}\right\} \longrightarrow \quad E_{S,f}[E] := \int_S \tilde{E}_i f^i \qquad \text{“flux through } S\text{”}$$

Loop quantum gravity: Take the $h_e$, $E_{S,f}$ as basic variables. Quantization should promote them to well defined operators.

## Quantization

Idea: Configuration space = space of connections $=: \mathcal{A}$. Then:

$$\mathcal{H} := L^2(\mathcal{A}, d\mu)$$

$$\widehat{A^i_a(p)}\Psi[A] = A^i_a(p)\Psi[A], \qquad \widehat{E^a_i(p)}\Psi[A] = \frac{\delta}{i\,\delta A^i_a(p)}\Psi[A]$$

But how to make that precise? By just thinking in terms of holonomies!
Some definitions:

**Graph** $\gamma$: Set of oriented paths ("edges") in $\Sigma$ intersecting at most in their endpoints ("vertices").

**Cylindrical function on** $\gamma$: Functional on $\mathcal{A}$ just depending on holonomies along edges of $\gamma$:

$$F[A] = f(h_{e_1}[A], h_{e_2}[A], \ldots), \quad e_1, e_2, \ldots \text{ edges of } \gamma, \quad f \in C^\infty(\mathrm{SU}(2)^{|\gamma|})$$

$\mathrm{Cyl}_\gamma$: Set of continuous functions cylindrical on $\gamma$.

Poisson bracket ($F \in \mathrm{Cyl}_\gamma \longleftrightarrow f \in C^\infty(\mathsf{SU}(2)^{|\gamma|})$):

$$\{F, E_{S,r}\} = 8\pi\, G \sum_{p \in S\cap\gamma} \sum_{\text{edges } e \text{ at } p} (\pm) r_i(p) \left(X^{R/L}\right)_e^i [f] =: 8\pi\, G X_{S,r}[F].$$

Throw all $\mathrm{Cyl}_\gamma$ together:

$$\mathrm{Cyl} := \left[\bigcup_\gamma \mathrm{Cyl}_\gamma\right] / \sim$$

Closing $\mathrm{Cyl}$ with respect to a natural norm gives C*-algebra $\overline{\mathrm{Cyl}} \cong C^0(\overline{\mathcal{A}})$.

What is the meaning of $\overline{\mathcal{A}}$? Can show that it's the set of **all** grupoid morphisms $\alpha$,

$$\alpha : \text{pathes in } \Sigma \quad \longrightarrow \quad \mathsf{SU}(2)$$

I.e. elements of $\overline{\mathcal{A}}$ are **generalized connections**.

$\overline{\mathcal{A}}$ can be given a measure by defining

$$\mathrm{Cyl}_\gamma \ni F : \qquad \int_{\overline{\mathcal{A}}} F \, d\mu_{\mathsf{AL}} := \int_{\mathsf{SU}(2)^{|\gamma|}} f(g_1, \ldots, g_{|\gamma|}) \, d\mu_{\mathsf{Haar}}$$

This measure is diffeomorphism invariant.

The measure is the basic ingredient for the quantization. It defines a Hilbert space

$$\mathcal{H}_{\mathsf{kin}} := L^2(\overline{\mathcal{A}}, d\mu_{\mathsf{AL}}).$$

which carries a representation of the basic variables:

Holonomies are cylindrical functions and act as multiplication operators:

$$\widehat{h}_e \Psi[A] = h_e[A]\Psi[A]$$

Fluxes act via invariant vector fields $X^R$, $X^L$ on $SU(2)$: $\psi_\gamma \in \mathrm{Cyl}_\gamma$

$$\widehat{E}_{S,f}\psi_\gamma = 8\pi i\, l_P^2 X_{S,f}[\psi_\gamma]$$

This representation has special status: It is the **unique** diffeomorphism invariant representation of the algebra of basic variables.

## Geometric Operators

Operators for *interesting* geometric quantities (Area, volume, angle. . . ) have been defined.

**Example**: Area operator. Classically

$$A(S) = \iota \int_s \sqrt{E \cdot E}$$

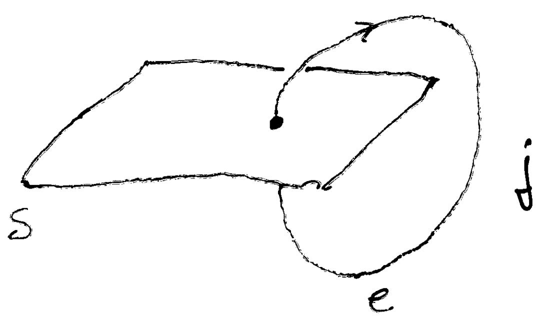
In quantum theory, define $\widehat{A}_S$ using the $E_{S,f}$ and point splitting.

Result: For $\psi_\gamma \in \mathrm{Cyl}_\gamma$:

$$\widehat{A}_S \psi_\gamma = 8\pi\iota l_P^2 \sum_{p \in S \cap \gamma} \sum_{\text{edges } e \text{ at } p} \sqrt{X_e \cdot X_e}\, \psi_\gamma .$$

Example revisited:

$$\widehat{A}_S \operatorname{Tr}(\pi_j(h_e[A])) = 8\pi l_P^2 \iota \sqrt{j(j+1)} \operatorname{Tr}(\pi_j(h_e[A]))$$



## Constraints

Implementation of constraint $C$ means:

1. Render $C$ as operator $\hat{C}$.
2. Just keep states $\Psi$ with $\hat{C}\Psi = 0$

Implementing the constraints for loop quantum gravity:

1. Gauss constraint: Easy. Gauge invariant states $\subset L^2(\overline{\mathcal{A}}, d\mu_{AL})$.
2. Diffeomorphism constraint: Feasible. Invariant states in the dual of $L^2(\overline{\mathcal{A}}, d\mu_{AL})$.
3. Hamiltonian constraint: Hard. "Problem of time". Proposal by Thiemann. Still under debate.

## Gauge invariant Hilbert space

Solutions to the Gauss constraint turn out to form subspace $\mathcal{H}_{\mathsf{GI}} \subset \mathcal{H}_{\mathsf{AL}}$.

States in $\mathcal{H}_{\mathsf{GI}}$ not only invariant under smooth gauge transformations but under generalized ones:

$$\mathcal{H}_{\mathsf{GI}} = L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, d\mu_{\mathsf{AL}}) = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_{\mathsf{AL}})$$

where $\overline{\mathcal{G}} = \{g : \Sigma \longrightarrow \mathrm{SU}(2)\}$ as a set.

An elegant basis is of $\mathcal{H}_{\mathsf{GI}}$ is given by **spin networks**. Data:

- graph $\gamma$
- for each edge $e$ in $\gamma$ half-integer $j_e$
- for each vertex $v$ in $\gamma$ intertwiner $I_v$ between tensor product of $j_e$ reps of SU(2) over all $e$ beginning in $v$ and tensor product of $j_e$ reps of SU(2) over all $e$ ending in $v$

Receipe to get $\Psi_{\gamma,\vec{j},\vec{I}}$: Take holonomies along edges in the rep of SU(2) corresponding to the $j$'s associated to the edges. Contract indices with the intertwiners in the obvious fashion.

## Diffeo invariant Hilbert space

Diffeomorphisms act unitarily on $\mathcal{H}_{\mathrm{GI}}$:

$$U(\varphi)\mathbf{\Psi}_{\gamma,\vec{\jmath},\vec{I}} = \mathbf{\Psi}_{\varphi(\gamma),\vec{\jmath},\vec{I}}$$

Solutions to the diffeomorphism constraint turn out to lie in the dual of $\mathcal{H}_{\mathrm{GI}}$:

$$\mathcal{H}_{\mathrm{GI}} \ni \psi \mapsto [\psi] \in \mathcal{H}_{\mathrm{diff}}, \qquad [\psi](\phi) = \sum_{\varphi:\mathrm{diffeo}} \langle \psi, U(\varphi)\phi \rangle$$

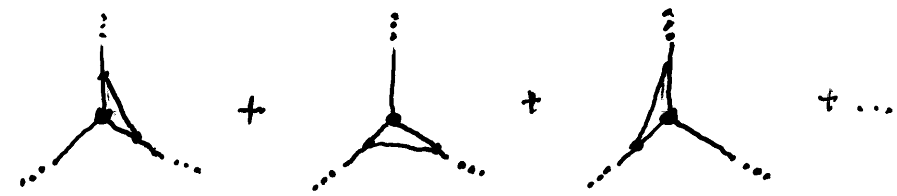
A scalar product on $\mathcal{H}_{\mathrm{diff}}$ is defined by $\langle [\psi], [\phi] \rangle = [\psi](\phi)$.

A convenient basis is given through spin networks $\mathbf{\Psi}_{\gamma,\vec{\jmath},\vec{I}}$ by $c_{\gamma,\vec{\jmath}}[\mathbf{\Psi}_{\gamma,\vec{\jmath},\vec{I}}]$

## Implementation of Hamilton constraint

Hamiltonian can be quantized on $\mathcal{H}_{\text{diff}}$ (Thiemann,Pullin + Gambini). $\widehat{H}(N)$ creates (and destroys) edges:

$$\widehat{H}(N)[\Psi_{\gamma,\vec{j},\vec{I}}] = \sum_{\text{vertex v}}$$



Kernel of $\widehat{H}$ should give $\mathcal{H}_{\text{phys}}$. Problems with $\widehat{H}(N)$:

- interpretation of solutions
- quantization ambiguities
- how compute scalar product in $\mathcal{H}_{\text{phys}}$ in practice?

## Spin foams

Idea:

$$\langle \Psi_{\gamma,\vec{j},\vec{I}}, \Psi_{\gamma',\vec{j}',\vec{I}'} \rangle_{\text{phys}} = \langle \Psi_{\gamma,\vec{j},\vec{I}}, \prod_x \delta(\widehat{H}(x)) \Psi_{\gamma',\vec{j}',\vec{I}'} \rangle_{\text{diff}}$$

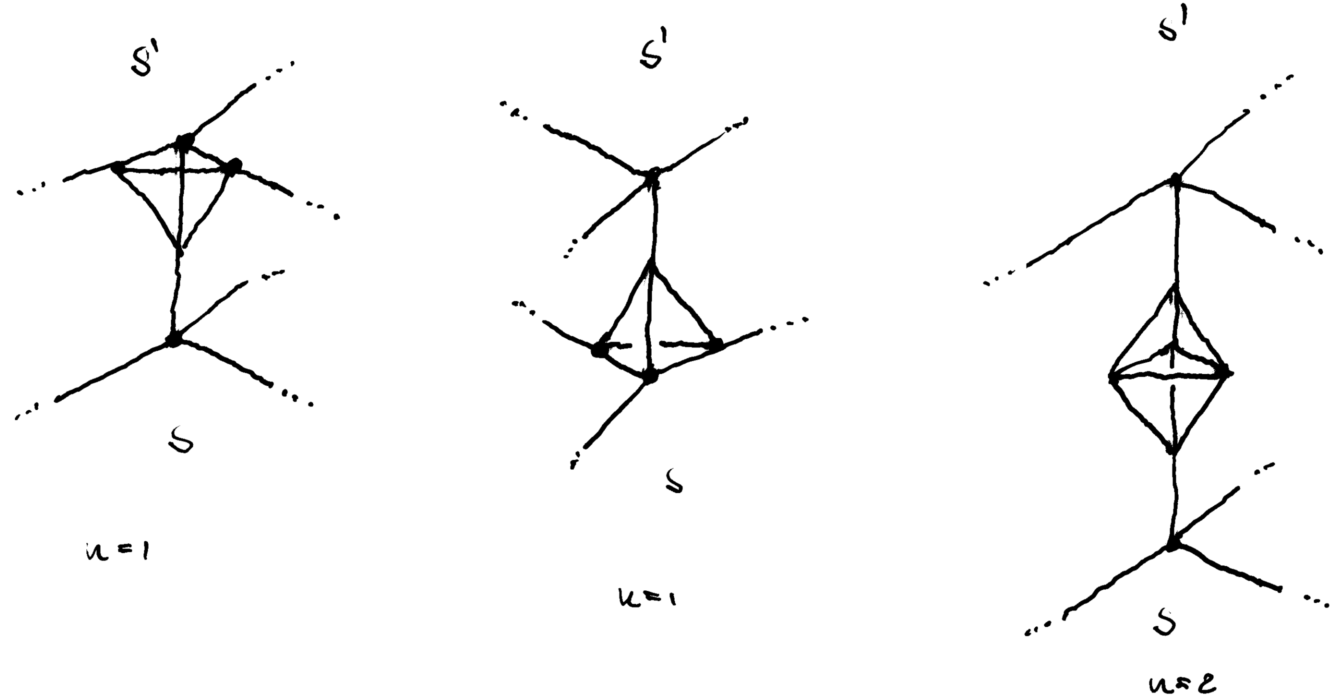
$$= \int \langle \Psi_{\gamma,\vec{j},\vec{I}}, \exp(i\widehat{H}(N)) \Psi_{\gamma',\vec{j}',\vec{I}'} \rangle_{\text{diff}} \, DN$$

Can rigorously define $DN$ for functions $|N| < T$, some $T$. Get

$$\langle S, S' \rangle_{\text{phys}} = \sum_{n=0}^{\infty} \frac{(iT)^n}{n!} \sum_{i_1,\ldots i_n,\alpha_1,\ldots\alpha_n} F(S,S')^{\alpha_1,\ldots\alpha_n}_{i_1,\ldots i_n} \langle S, S'^{\alpha_1,\ldots\alpha_n}_{i_1,\ldots i_n} \rangle_{\text{diff}}$$

The terms in the sum can be labeled by histories of spin networks, spin foams. They come with an amplitude: Analogy with Feynman diagramms.

Some examples:




Spin foams are also obtained from regularizing the 4d path integral (spin-foam models). This gives possibilities to compare covariant and canonical approach!

## Sketch of BH entropy calculation

Can not yet treat black holes from first principles. But can do LQG for space times $M$ that have an **isolated horizon** as inner boundary.

Isolated horizon: Local notion, lightlike 3 surface $H \cong S^2 \times \mathbb{R}$, conditions on geometry such that no gravitational radiation falls in or comes out.

Natural action on such $M$ from which Einsteins equations follow. Action includes surface term on $H$ wich gives "laws of IH mechanics" analogous to the laws of BH mechanics.

In connection variables, the surface term looks like a U(1) Chern-Simons action with level

$$k = \frac{A_0}{4\pi\gamma l_P^2}$$

Canonical formulation available in terms of $(A, E)$ on $\Sigma$ with boundary 2-sphere $S$ as before, U(1) connection given by $W_a = \Gamma_a^i|_S r_i$ (with $r : S \longrightarrow$ su(2)), and boundary conditions are

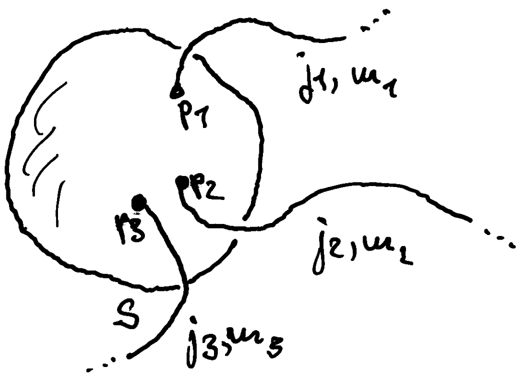
$$F_{ab}(W) = -\frac{2\pi\gamma}{A_0} E_i^c \epsilon_{cab} r^i.$$

Working with generalized connections: $\mathcal{H}_{\mathsf{kin}} \subset \mathcal{H}_S \bigotimes \mathcal{H}_V$.

Eigenstates for $\widehat{A}_S$, $\widehat{E_{S,r}}$ in $\mathcal{H}_V$:

$$\widehat{A}_S \Psi_{\vec{p},\vec{j},\vec{m}} = 8\pi l_P^2 \gamma \sum_I \sqrt{j_I(j_I+1)}\Psi_{\vec{p},\vec{j},\vec{m}},$$

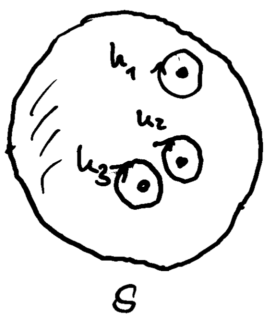
$$\widehat{E_{S,r}} \Psi_{\vec{p},\vec{j},\vec{m}} = 8\pi l_P^2 \sum_I m_I \Psi_{\vec{p},\vec{j},\vec{m}}$$






Eigenstates for holonomies $h_I$ around points $p_I$ in $\mathcal{H}_S$:

$$h_I \phi_{\vec{p},\vec{a}} = \exp \frac{2\pi i a_I}{k} \phi_{\vec{p},\vec{a}}, \qquad a_I \in \mathbb{Z}_k, \quad \sum_I a_I = 0$$



Boundary conditions put constraint on vectors in $\mathcal{H}_{\text{kin}}$:

$$F_{ab}(W) = -\frac{2\pi\gamma}{A_0} E_i^c \epsilon_{cab} r^i .$$

translates to

$$\exp(i\widehat{F})\phi_S \otimes \Psi_V = \phi_S \otimes \exp\left(-i\frac{2\pi\gamma}{A_0}\widehat{E}_{S,r}\right)\Psi_V$$

Can be solved only if spectra of operators involved are same. They are!

$\mathcal{H}_{\text{kin}}$ is spanned by vectors

$$\phi_{\vec{p},\vec{a}} \otimes \Psi_{\vec{p'},\vec{j},\vec{m}} \quad \text{with} \quad p = p' \text{ and } 2m_I = -a_I \mod k$$

Remains to implement the constraints:

1. Gauss constraint: on $S$ already taken care of, easy away from S.
2. Diffeomorphism constraint: Identify states that are transformed into each other under a diffeo.
3. Hamiltonian constraint: Not really relevant, because action on $S$ is trivial. Assumption: To each set of data $\{\vec{j}, \vec{m}\}$ find at least one solution that is eigenvector of $\widehat{A}_S$, $\widehat{E_{S,r}}$ with the eigenvalues given by the data.

Now one can can count: $N_{A_0} =$ number of data $\{\vec{a}, \vec{m}, \vec{j}\}$ such that

- $2m_I = -a_I \mod k$
- $8\pi l_P^2 \gamma \sum_I \sqrt{j_I(j_I+1)} \in [A_0 - l_P^2, A_0 + l_P^2]$.

Result for large $A_0$ is

$$S = \ln N_{A_0} = \frac{\ln 2}{4\pi\sqrt{3}\gamma l_P^2} A_0 + \text{logarithmic corrections.}$$

Comparison with Bekenstein-Hawking formula gives $\gamma = \ln 2/\pi\sqrt{3}$.

Analysis goes through for GR coupled to Maxwell, Yang Mills fields, etc. and gives same formula and hence value for $\gamma$.

## Summary

Hope to have convinced you that LQG is interesting:

- relatively simple
- gives quantized spacial geometry
- nice mathematical formalism for treating path integrals over connections

Already some applications:

- black hole entropy
- loop quantum cosmology

But there are lots of things that still have to be fully understood:

- implementation of Hamiltonian, problem of time, observables
- semiclassical states
- specific predictions that might be testable
- connection to other approaches