Operator vs. Geometric approach to String Field Theory

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based on:

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and work with I. Ellwood

Plan:

1. A bit of physics - Sen's conjectures

2. Formal structure

3. Surface states
   - wedge states, butterflies, projectors

4. Conservation laws

5. Product of regularized butterflies

6. Open problems

Sen's conjectures

Open bosonic SFT $\Leftrightarrow$ theory of D25 brane
Tachyon $\rightarrow$ instability of the pert. vacuum
New stable vacuum $\rightarrow$ NO open strings

- The difference between the energy of the unstable and the perturbatively stable vacuum state ΔE is equal to the rest mass $T_{25}V_{25}$ of the unstable D25-brane.

- There are translationally noninvariant vacua which correspond to the lower-dimensional branes. They are lump configurations in the tachyon and other string fields with exactly the right energies to be interpreted as D-branes.

- The perturbatively stable vacuum is the closed string vacuum only and hence there should be no open string excitations around.

String Field Theory

(Witten's open bosonic cubic SFT)

Standard string theory: calculates on-shell amplitudes by putting vertex operators on the worldsheet

String field theory:

1) is a field theory for all the component fields

2) gives natural off-shell extension for string amplitudes

3) decomposes string diagrams into vertices and propagators

Various formulations:

1) Schrödinger representation (Witten)

2) Fock representation (Gross, Jevicki)

3) CFT approach (LeClair, Peskin, Preitschopf)

Witten's formulation

Basic object is the string field

$$|\Psi\rangle = \int dp \left[t(p) c_1 |0\rangle + A_\mu(p) \alpha^\mu_{-1} c_1 |0\rangle + \cdots \right]$$

The string field action has form of noncommutative Chern-Simons action

$$S = -\frac{1}{g^2} \int \left(\frac{1}{2} \Psi * Q\Psi + \frac{1}{3} \Psi * \Psi * \Psi \right)$$

where

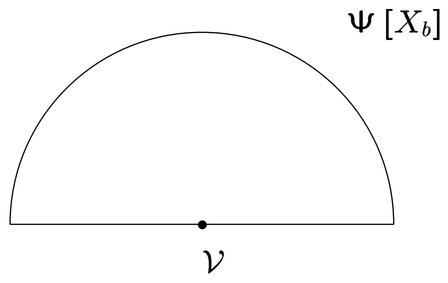
$$\begin{aligned} Q^2 &= 0, \\ \int Q\Psi &= 0, \\ Q(A * B) &= QA * B + (-)^A A * QB \end{aligned}$$

Nice feature is manifest gauge invariance

$$\delta\Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi,$$

In Schrödinger representation

$$\Psi\,[X(\sigma), c(\sigma)] = \langle\, X(\sigma), c(\sigma) | \Psi \,\rangle$$

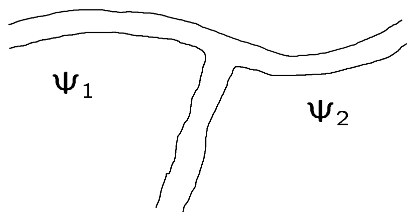


The star product is defined by

$$(\Psi_1 * \Psi_2)[X_0(\sigma)] = \int \mathcal{D}X_1^R \mathcal{D}X_2^L\, \delta\left[X_1^R - X_2^L\right] \Psi_1[X_1(\sigma)]\Psi_2[X_2(\sigma)]$$

where

$$\begin{aligned} X_1^L(\sigma) &= X_0^L(\sigma) \\ X_2^R(\sigma) &= X_0^R(\sigma) \end{aligned}$$

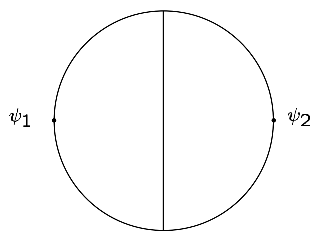


CFT approach

(LeClair, Peskin, Preitschopf)

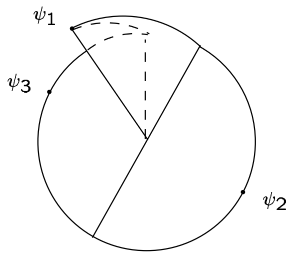
Two vertex can be obtained by gluing two half-disks

$\langle \psi_1, \psi_2 \rangle =$



Three vertex can be obtained by gluing three half-disks

$\langle \psi_3, \psi_1 * \psi_2 \rangle =$



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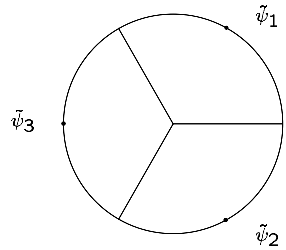


Let us put pictures into formulas:

The n-point vertex is

$$\langle\, \Psi_1, \Psi_2, \ldots, \Psi_n \,\rangle = \langle\, f_1^{(n)} \circ \Psi_1(0) \; \ldots \; f_n^{(n)} \circ \Psi_n(0) \,\rangle,$$

which we write also as

$$\begin{aligned} \langle\, \Psi_1, \Psi_2, \ldots, \Psi_n \,\rangle &= \int \Psi_1 * \Psi_2 * \cdots * \Psi_n \\ &= \langle V_{12\ldots n} || \Psi_1 \rangle \otimes \cdots \otimes |\Psi_n \rangle . \end{aligned}$$

where

$$\begin{aligned} h(z) &= \frac{1+iz}{1-iz} \\ f_k^{(n)} &= h^{-1}\left(e^{-\frac{2\pi i k}{n}} (h(z))^{\frac{2}{n}} \right) \end{aligned}$$

Fock $\Leftrightarrow$ CFT approach

For the 3-point correlator we need

$$f \circ a_{-n} = \oint \frac{dz}{2\pi i} z^{-n} f'(z) \partial_z X(f(z))$$

Contractions of these operators are

$$f_r \circ [...a_{-n}...] \, f_s \circ [...a_{-m}...] =$$
$$= \oint \frac{dz}{2\pi i} z^{-n} f_r'(z) \oint \frac{dw}{2\pi i} w^{-m} f_s'(w) \frac{-1}{(f_r(z) - f_s(w))^2}$$

Their effect is that of

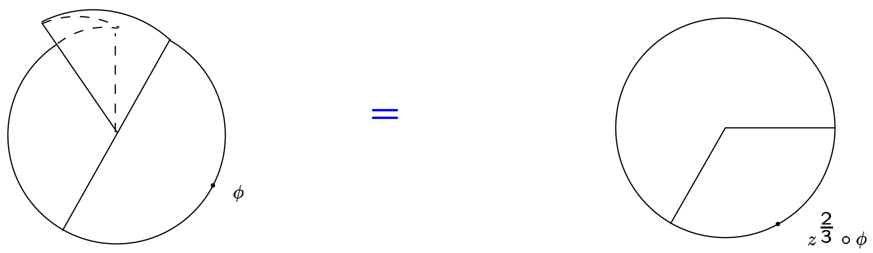
$$\langle V_{12...n}| = \langle 0| \otimes \langle 0| \otimes \cdots \otimes \langle 0| \, e^{-\frac{1}{2}\sum_{r,s} \sum_{m,n\geq 1} a_n^r N_{nm}^{rs} a_m^s}$$

with the Neumann coefficients

$$N_{nm}^{rs} = \frac{-1}{nm} \oint \frac{dz}{2\pi i} z^{-n} \oint \frac{dw}{2\pi i} w^{-m} \frac{f_r'(z) f_s'(w)}{(f_r(z) - f_s(w))^2}.$$

Surface states

Star product of two vacua is a state which looks like this:



In more generality we can associate a (bra) state to any disk with a choice of local coordinate. Those states are defined by

$$\langle f|\phi\rangle = \langle f \circ \phi\rangle, \qquad \forall\phi$$

Conformal maps $f(z)$ are represented on the Hilbert space by the operators U_f

$$f \circ \Psi = U_f \Psi(z) U_f^{-1}$$

For a primary field of dimension d

$$f \circ \Psi(z) = \left[f'(z)\right]^d \Psi(f(z))$$

These operators can be realized as

$$U_f = e^{\sum v_n L_n}$$

where $v(z) = \sum v_n z^{n+1}$ and

$$v(z)\partial_z f(z) = v(f(z))$$

Therefore

$$\langle f| = \langle 0|U_f$$

Wedge states

The wedge states are surface states associated to the conformal map

$$f(z) = h^{-1}\left(h(z)^{2/r}\right) = \tan(2/r \arctan z)$$

The map f is generated by a vector field

$$v(z) = (1 + z^2) \arctan z$$

times a constant $\log(2/r)$. Define:

$$\begin{aligned}
\mathcal{A} &= \oint \frac{dz}{2\pi i} v(z) T(z) = L_0 + \frac{2}{3} L_{-2} - \frac{2}{15} L_{-4} + \cdots, \\
\mathcal{A}^\dagger &= \oint \frac{dz}{2\pi i} v(z) I \circ T(z) = L_0 + \frac{2}{3} L_2 - \frac{2}{15} L_4 + \cdots, \\
K_1 &= \oint \frac{dz}{2\pi i} (1 + z^2) T(z) = L_1 + L_{-1}.
\end{aligned}$$

These objects obey the algebra

$$\begin{aligned}
\left[\mathcal{A}, \mathcal{A}^\dagger\right] &= \mathcal{A} + \mathcal{A}^\dagger, \\
[\mathcal{A}, K_1] &= K_1, \\
\left[\mathcal{A}^\dagger, K_1\right] &= -K_1.
\end{aligned}$$

The operator U_f is simply

$$U_r = (2/r)^{\mathcal{A}}$$

Explicitly the wedge states are

$$|r\rangle = (2/r)^{\mathcal{A}^\dagger}|0\rangle$$

The commutation relations can be used to find

$$U_r U_s^\dagger = U^\dagger_{\frac{2}{r}(r+s-2)} U_{\frac{2}{s}(r+s-2)}$$

With some additional algebra one can find the star product of wedge states

$$|r\rangle * |s\rangle = |r+s-1\rangle.$$

purely algebraically! This law is obvious in the geometric picture.

We get an interesting identity:

$$\tan\left(\frac{2}{r}\arctan\left(\cot\left(\frac{s}{2}\operatorname{arccot} z\right)\right)\right) =$$
$$\cot\left(\frac{r+s-2}{r}\operatorname{arccot}\left(\tan\left(\frac{s}{r+s-2}\arctan z\right)\right)\right)$$

Note: The star algebra derivation K_1 annihilates all wedge states, in particular the limiting state called sliver which is a projector.

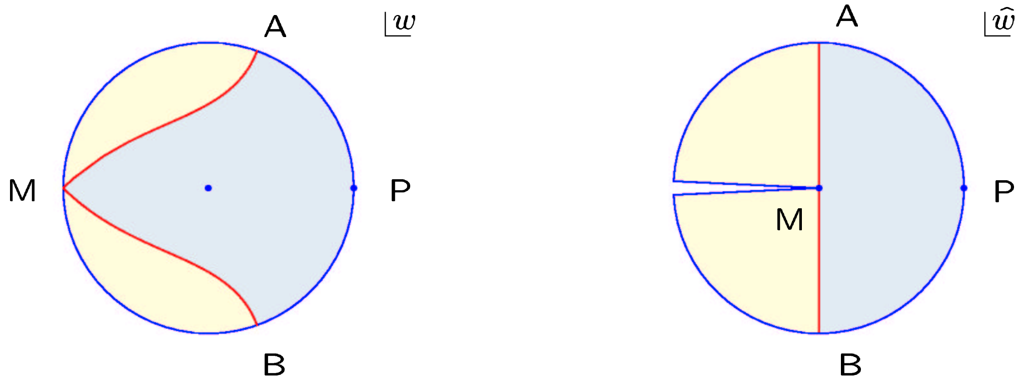
Butterfly states

The butterfly states are surface states associ-ated to the conformal map

$$
\begin{aligned}
f_r(z) &= \sin\left(\frac{2}{r}\arctan z\right) \\
&= \frac{z}{\sqrt{1+z^2}} \circ \tan\left(\frac{2}{r}\arctan z\right)
\end{aligned}
$$

The simplest example is

$$|B\rangle = e^{-\frac{1}{2}L_{-2}}|0\rangle$$



Two special features:
1) projector arising from numerical calculations in VSFT
2) it is annihilated by $K_2 = L_2 - L_{-2}$.

Algebraic proof that butterflies are projectors: write

$$|B\rangle = \lim_{\alpha\to\infty} e^{\alpha K_2}|0\rangle$$

and use two simple identities

$$e^{\alpha K_2} = e^{-\frac{1}{2}\tanh 2\alpha\, L_{-2}}\,(\cosh 2\alpha)^{-L_0}\, e^{\frac{1}{2}\tanh 2\alpha\, L_2},$$

$$e^{\alpha K_2}(|\phi\rangle * |\chi\rangle) = e^{\alpha K_2}|\phi\rangle * e^{\alpha K_2}|\chi\rangle,$$

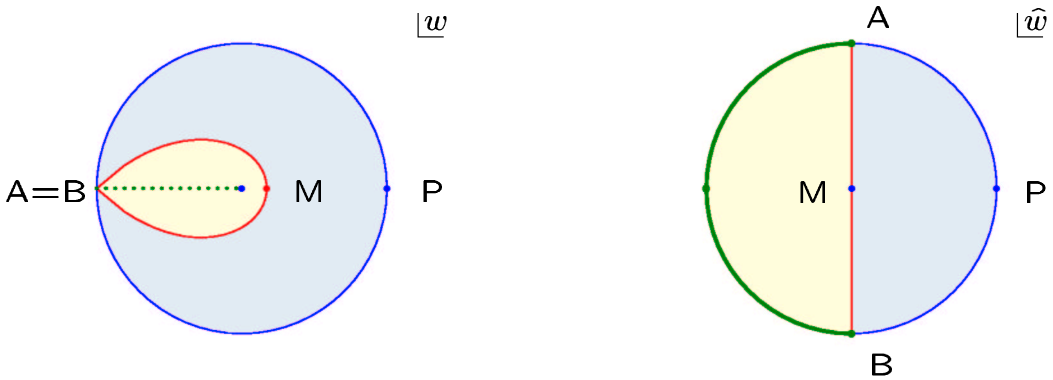
Observe that for $\alpha \to +\infty$

$$e^{\alpha K_2}|3\rangle \approx e^{\alpha K_2}|0\rangle$$

It remains to show that

$$\lim_{\alpha\to\infty} e^{-\frac{1}{2}\tanh 2\alpha\, L_2} U_3^\dagger|0\rangle = e^{-\frac{1}{2}\, L_2} U_3^\dagger|0\rangle$$

is finite and unambiguous. We need a geometric argument



The surface is regular in the neighborhood of the local patch. Can be glued onto wedge surfaces.

Conservation laws

In general the Virasoro conservation laws can be derived as

$$0 = \langle 0| \oint v(z) T(z) U_f = \langle f| \oint v(z) f^{-1} \circ T(z)$$

for $v(z)$ holomorphic around infinity.

For the butterfly the laws are very simple:

$$\begin{aligned}
(L_0 - 2sL_{-2}) e^{sL_{-2}} |0\rangle &= 0, \\
(L_2 - (2s)^2 L_{-2}) e^{sL_{-2}} |0\rangle &= \frac{cs}{2} e^{sL_{-2}} |0\rangle, \\
(L_{2k} - (2s)^{k+1} L_{-2}) e^{sL_{-2}} |0\rangle &= \frac{c(k+1)}{8} (2s)^k e^{sL_{-2}} |0\rangle, \qquad k \geq 1.
\end{aligned}$$

Product of regularized butterflies

We have to rewrite the operator

$$e^{sL_2}\, U_3^{\dagger}|0\rangle$$

in a normal ordered form. It is again surface state. Find the associated conformal map. Let us start with a solution to a related problem: Summing the conservation laws we find

$$U_r\, e^{\frac{1}{2}\tan^2\gamma L_{-2}}|0\rangle = \mathcal{N}^c\, e^{\frac{1}{2}\tan^2\frac{2\gamma}{r}L_{-2}}|0\rangle$$

In general for any two maps f, g we write

$$U_f U_g^{\dagger} = U_{\phi[f,g]}^{\dagger} U_{\phi[g,f]}$$

so we have

$$f \circ I \circ g^{-1} \circ I = I \circ \phi^{-1}[f, g] \circ I \circ \phi[g, f]$$

In our case

$$f(z) = \tan\left(\frac{2}{r}\arctan z\right), \qquad g(z) = \frac{z}{\sqrt{1-\tan^2\gamma\, z^2}}$$

and we find

$$\phi[f,g](z) = \frac{z}{\sqrt{1-\tan^2\left(\frac{2\gamma}{r}\right)\, z^2}}$$

$$\phi[g,f](z) = \sqrt{\tan^2\left(\frac{2}{r}\arctan\sqrt{z^2+\tan^2\gamma}\right) - \tan^2\frac{2\gamma}{r}}.$$

Putting things together

$$e^{-\frac{1}{2}\tan^2\gamma L_{-2}}|0\rangle * e^{-\frac{1}{2}\tan^2\gamma L_{-2}}|0\rangle = U^\dagger_{f_{3,\gamma}}|0\rangle$$

is a surface state associated to the map

$$f_{3,\gamma}(z) = \sqrt{\tan^2\left(\frac{2}{3}\arctan\sqrt{\frac{z^2+\tan^2\gamma}{1+z^2\tan^2\gamma}}\right) - \tan^2\frac{2\gamma}{3}}$$

and therefore it looks as

$$U^\dagger_{f_{3,\gamma}}|0\rangle = e^{v_2 L_{-2}+v_4 L_{-4}+\cdots}|0\rangle,$$

where the first two coefficients are

$$v_2 = \frac{-3-4\cos\frac{4\gamma}{3}+2\cos\frac{8\gamma}{3}}{3\left(1+2\cos\frac{4\gamma}{3}\right)^2},$$

$$v_4 = \frac{\left(1-2\cos\frac{4\gamma}{3}\right)^2\left(47+60\cos\frac{4\gamma}{3}+10\cos\frac{8\gamma}{3}\right)}{54\left(1+2\cos\frac{4\gamma}{3}\right)^4}.$$

Note, that $h \circ f_{3,\gamma} \circ h^{-1}(z^{\frac{3}{2}})$ maps

Can't be found in the dictionary of conformal maps. Looks impossible to find it by conventional methods.

Why did the butterfly show up in the calculation of Gaiotto, Rastelli, Sen, Zwiebach ?

The product of two near-butterflies (i.e.$\gamma \sim \frac{\pi}{4}$) is a map

$$f(z) \sim \left[\left(1 + \frac{2}{\sqrt{3}} \left(\gamma - \frac{\pi}{4} \right) z \frac{d}{dz} \right) \frac{z}{\sqrt{1+z^2}} + O \left(\gamma - \frac{\pi}{4} \right)^2 \right]$$

Therefore

$$e^{sL_{-2}}|0\rangle * e^{sL_{-2}}|0\rangle = \left[1 + \left(1 - \frac{1}{\sqrt{3}} \right) \left(s + \frac{1}{2} \right) L_0 \right] e^{sL_{-2}}|0\rangle + O \left(s + \frac{1}{2} \right)^2$$

We thus see that $|\psi\rangle = e^{sL_{-2}}|0\rangle$ solves the equation of motion of ghost number zero string field theory

$$|\psi\rangle * |\psi\rangle = \left(1 + a^{-1} L_0 \right) |\psi\rangle$$

in the limit $a \to \infty$, provided we identify

$$a^{-1} = \left(1 - \frac{1}{\sqrt{3}} \right) \left(s + \frac{1}{2} \right) .$$

Open problems, further directions

1. Normal ordering of $U_f U_g^\dagger = U^\dagger_{\phi[f,g]} U_{\phi[g,f]}$ or alternatively decomposing

   $$f \circ I \circ g^{-1} \circ I = I \circ \phi^{-1}[f,g] \circ I \circ \phi[g,f]$$

   Note that for $f \equiv g$ the map

   $$I \circ \phi \circ I \circ f$$

   maps unit circle into itself. Finding ϕ given f is one of the classical problems of complex analysis.

2. Relation with dToda hierarchy, see Boyarsky and Ruchayskiy; Bonora and Sorin. Surface states are squeezed states

   $$\langle f| = \langle 0|\, e^{-\frac{1}{2}\sum_{m,n\geq 1} a_n N_{nm} a_m}$$

   Hirota identities put simple constraints on possible forms of N_{nm}. How are these constraints preserved under gluing ?

3. What can be done in Siegel gauge? Can we find ϕ corresponding to

$$f(z) = \tan(2/r \arctan \alpha z)$$

It seems that this is the very first problem one has to solve before any analytic progress can be done.

4. Other gauges? Gauges in which L_0 is replaced by combinations of $\mathcal{A}, \mathcal{A}^\dagger, K_1$ would lead to some simplifications.

5. Classification of projectors. Seems, that all surface state projectors are in 1-1 correspondence with derivations.

6. Once we develop the technology, we can start looking for closed strings, calculate off-shell and loop diagrams, study marginal deformations, rolling tachyons ...