Negative Energy and the Focussing of Light Rays

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references:

- “A Quantum Focussing Conjecture” (Raphael Bousso, Zach Fisher, Stefan Leichenauer, AW)
- “Proof of the Quantum Null Energy Condition” (ditto + Jason Koeller)
- “The Generalized Second Law implies a Quantum Singularity Theorem” (AW)
- “A Second Law for Higher Curvature Gravity” (AW)

Lightning Review of General Relativity

notation uses tensors to easily ensure coordinate invariance

basic field is the metric $g_{\mu\nu}$, a 4 x 4 symmetric tensor

Riemann curvature tensor $R_{\mu\nu o\xi}$ involves two derivatives of the metric, contracting indices using the inverse metric $g^{\mu\nu}$ gives the Ricci tensor $R_{\mu\nu}$ and scalar R.

Einstein Field Equation:

$$R_{\mu\nu} - (1/2) g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

curvature of spacetime

stress-energy tensor of matter fields

Stress-Energy Tensor

the stress-energy tensor is also a 4x4 symmetric matrix;

can be interpreted in a “local inertial coordinate system” (t, x, y, z) as:

$$T_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} \quad t \quad & \quad x \quad & \quad y \quad & \quad z \quad \end{array} \\ \left(\begin{array}{c|ccc} T_{tt} & T_{tx} & T_{ty} & T_{tz} \\ \hline \text{“} & T_{xx} & T_{xy} & T_{xz} \\ \text{“} & \text{“} & T_{yy} & T_{yz} \\ \text{“} & \text{“} & \text{“} & T_{zz} \end{array} \right) \end{array}$$

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$$
\begin{array}{cc}
 & \begin{array}{c|ccc} \quad t \quad & \quad x \quad & \quad y \quad & \quad z \quad \end{array} \\
T_{\mu\nu} = & \left(\begin{array}{c|ccc}
\begin{array}{l}\text{energy} \\ \text{density}\end{array} & \multicolumn{3}{c}{\begin{array}{c}\text{momentum-density} \\ \text{(= energy flux)}\end{array}} \\ \hline
\text{“} & \text{x-pressure} & & \textbf{stress} \\
\text{“} & \text{“} & \text{y-pressure} & \\
\text{“} & \text{“} & \text{“} & \text{z-pressure}
\end{array}\right)
\end{array}
$$

Perfect Fluids

special case: fluid with energy density ρ and pressure p,
(in rest frame)

$$T_{\mu\nu} = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} t & x & y & z \end{array}$$

$$T_{\mu\nu} = \left(\begin{array}{c|ccc} \rho & 0 & 0 & 0 \\ \hline 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array}\right)$$

Spacetime geometry is not fixed *a priori*
—what spacetimes are allowed?

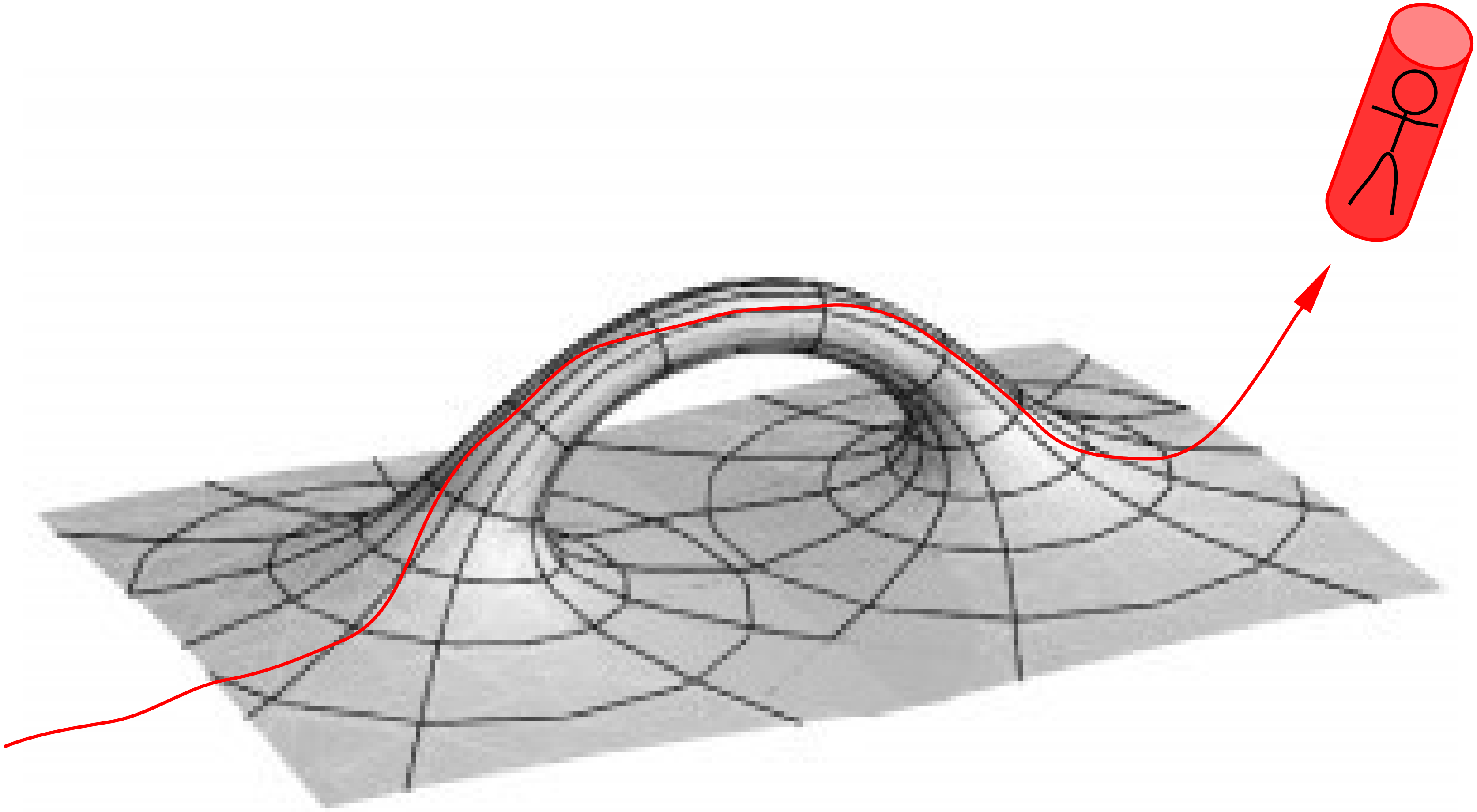
If there are no restrictions on $T_{\mu\nu}$,
Einstein's Equation has no content,
and *any* geometry you like could be a solution:

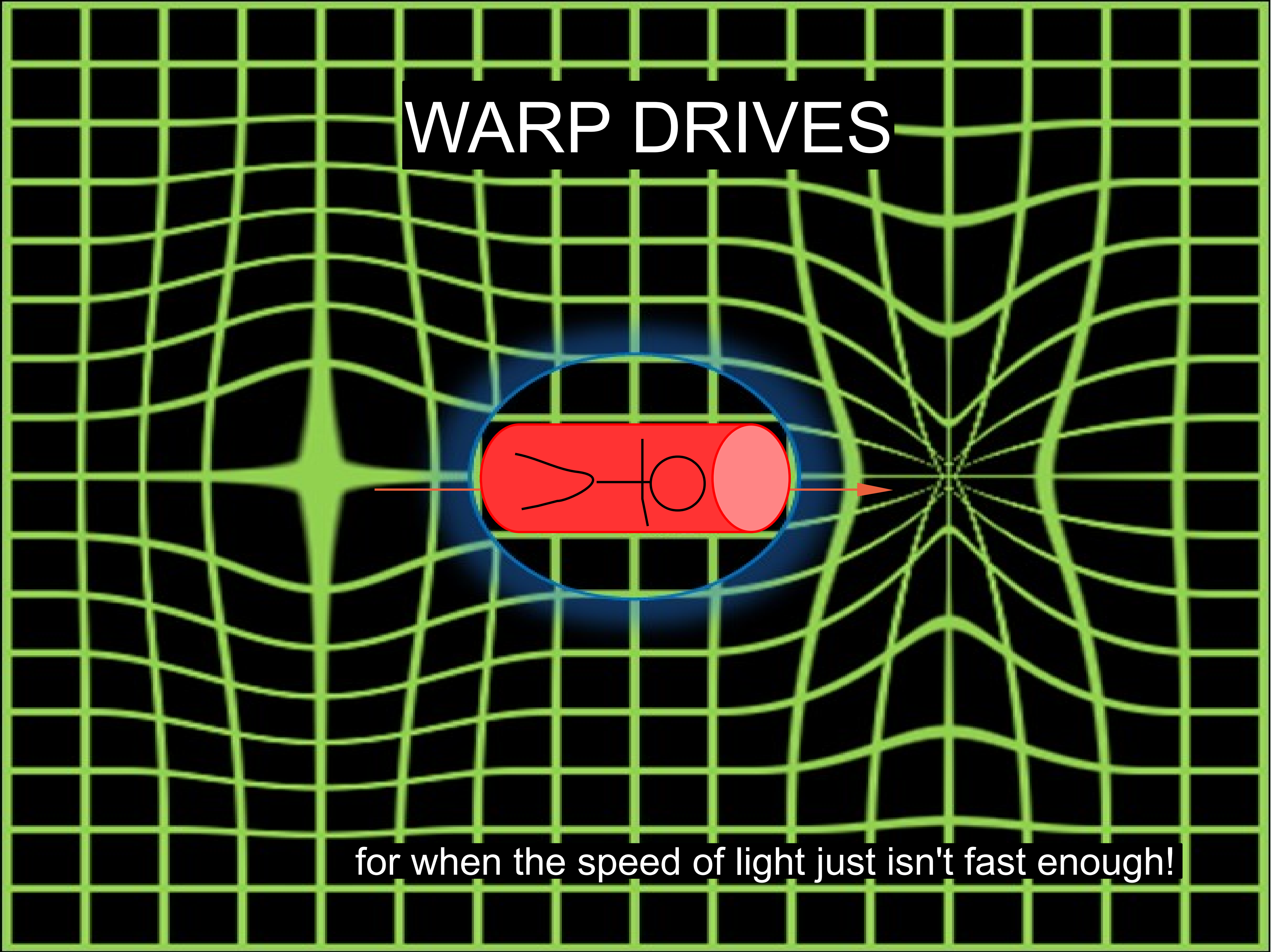
$$g_{\mu\nu} = ?$$

Many science fiction possibilities...

TRAVERSABLE WORMHOLES

for getting to another universe, or elsewhere in our own



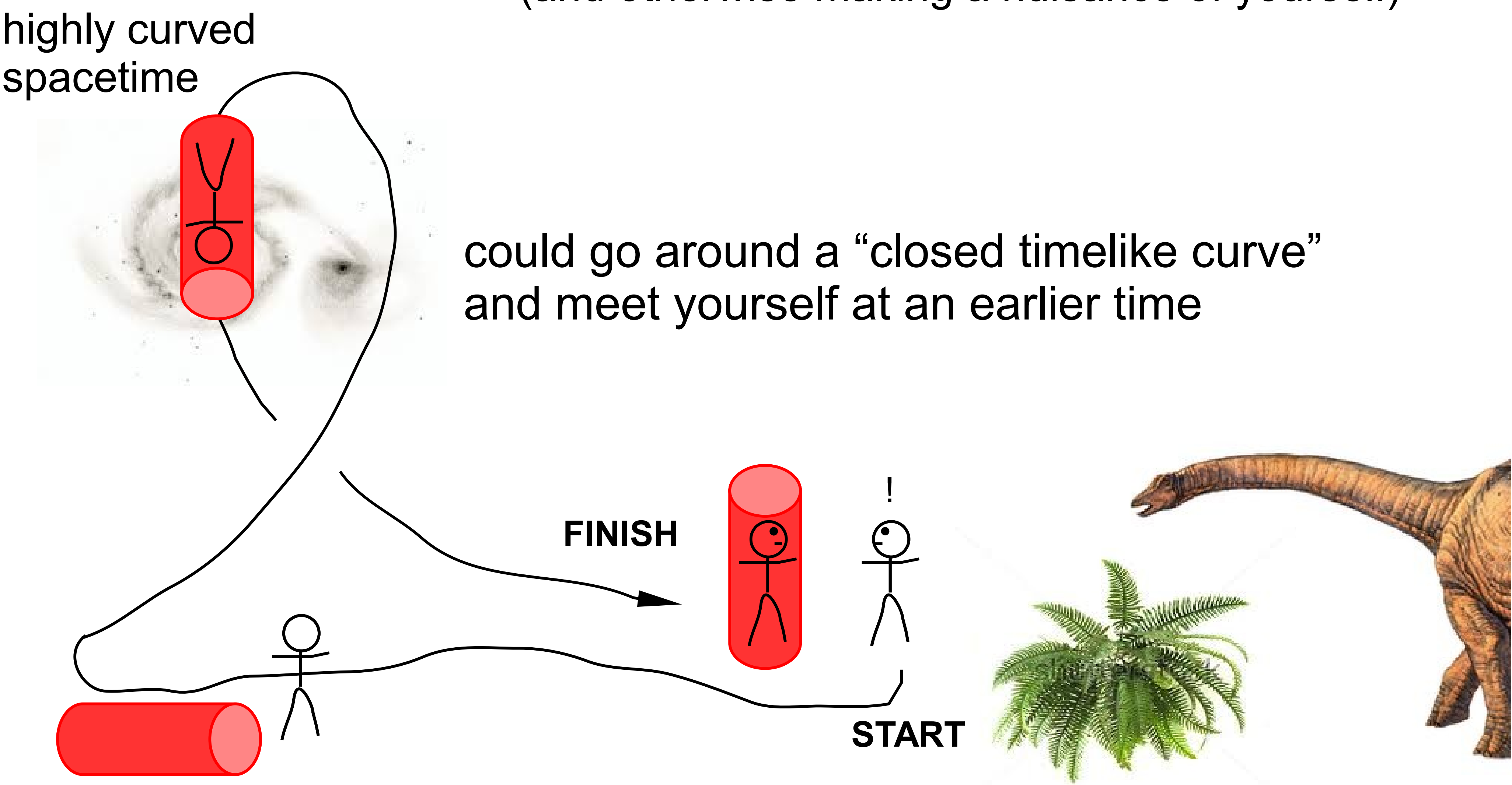

WARP DRIVES
for when the speed of light just isn't fast enough!

and worst of all:

TIME MACHINES

for killing your grandfather before you are born
(and otherwise making a nuisance of yourself)

could go around a “closed timelike curve”
and meet yourself at an earlier time



BUT ARE THESE CRAZY THINGS ACTUALLY POSSIBLE?

Probably not.*

All of them require exotic matter which violates some “energy condition” normally obeyed by reasonable fields.

*except actually maybe yes for traversable wormholes, see my recent paper with Daniel Jafferis & Ping Gao...

Some Energy Conditions

k^μ : null vector $\qquad t^\mu,\ u^\mu$: future timelike vectors

implies

Condition	*this can't be negative:*	*perfect fluid*	*interpretation*
Null	$T_{\mu\nu}\, k^\mu k^\nu$	$\rho + p \geq 0$	null surfaces focus
Weak	$T_{\mu\nu}\, t^\mu\, t^\nu$	$\rho \geq 0$ $\rho + p \geq 0$	positive energy in any frame
Dominant	$T_{\mu\nu}\, t^\mu\, u^\nu$	$\rho \geq \|p\|$	energy can't go faster than light
Strong	$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) t^\mu t^\nu$	$\rho + p \geq 0$ $\rho + 3p \geq 0$	timelike geodesics focus

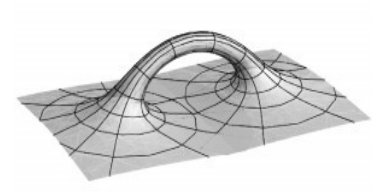
Strong energy condition is violated for scalar fields with potential $V(\phi)$, e.g. inflation

All of these conditions are violated by quantum fields!

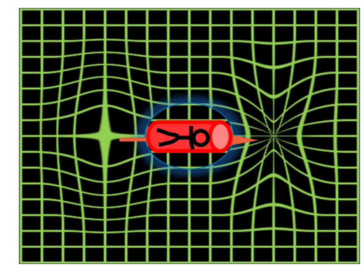
some classical GR theorems

using the ***null energy condition*** (plus technical auxilliary assumptions), one can show:

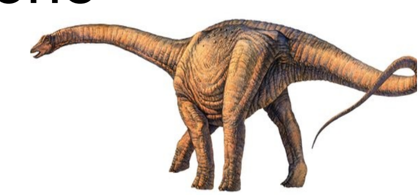
- No traversable wormholes (topological censorship)
  Morris-Thorne-Yurtsever (88), Friedman-Schleich-Witt (93)



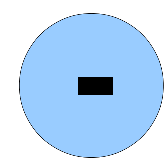
- No warp drives (from past infinity to future infinity)
  Olum (98), Gao-Wald (00), Visser-Bassett-Liberati (00)



- No time machines can be created if you start without one
  Tipler (76), Hawking (92)



- No negative mass isolated objects (Shapiro advance)
  Penrose-Sorkin-Woolgar (93), Woolgar (94), Gao-Wald (00)



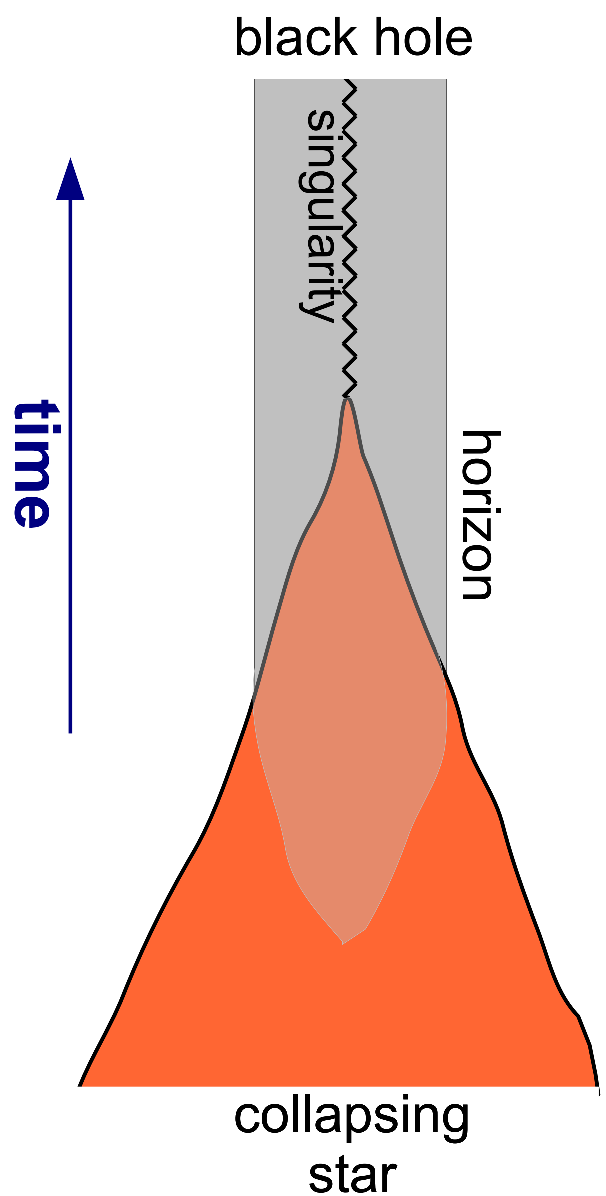
(although you need the ***dominant energy condition*** to prove that there can't be a negative energy bubble of “false vacuum” which travels outwards at the speed of light and destroys the universe!)

Positive energy theorem: Shoen-Yau (79) Witten (81)

There's another seemingly pathological feature of spacetimes in General Relativity...

and here the energy conditions won't help us, in fact they cause the problem...

Singularities



Classical general relativity predicts singularities, places where spacetime comes to an end and cannot be extended any further.

E.g. when a star collapses to form a black hole, there's a singularity (where time ends for an infalling observer) inside of the event horizon.

Also Big Bang singularity at beginning of time.

Singularity Theorems

these show that singularities form in certain *generic* situations. 2 main types:

1) The original *Penrose theorem* is based on showing that lightrays focus into a singularity in strong gravitational situations (e.g. black holes) so it requires the ***null energy condition****.

Hawking used it to prove a Big Bang singularity, but only if our universe is open (flat or hyperbolic).

2) The *Hawking theorem(s)* show that timelike rays converge to a singularity, so it uses the ***strong energy condition****. Works for closed spacetimes, but SEC is untrue e.g. during inflation...

(Borde-Guth-Vilenkin theorem says that inflation had to have a beginning, often *called* a singularity theorem but quite different, e.g. no energy condition)

*****plus technical assumptions

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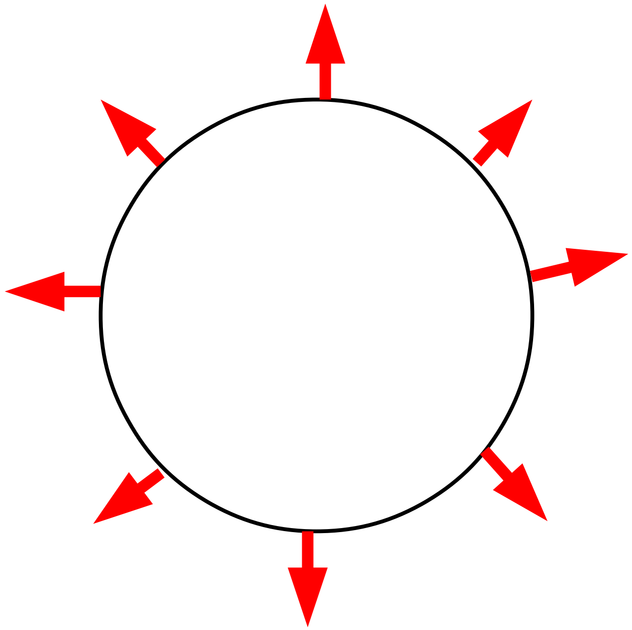
Penrose Singularity Theorem

Theorem of classical GR. Penrose (65).

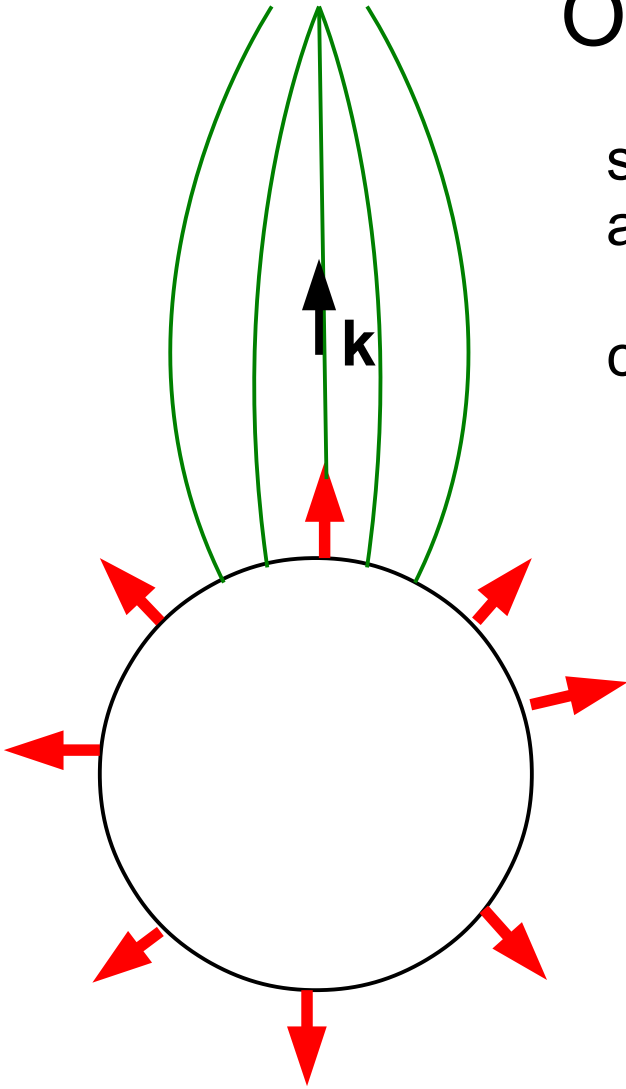
Assumes 1. null energy condition ($T_{kk} \geq 0$, k is null)
2. global hyperbolicity
3. space is infinite

Says that IF a trapped surface forms, then a singularity is inevitable.

A trapped surface is a closed (D-2)-dimensional surface for which the expansion of outgoing null rays is negative.
(i.e. area is decreasing everywhere)



Outline of Penrose Proof



shoot out lightrays from the null surface...
attractive gravity causes lightrays to focus!

calculate focusing w/ Raychaudhuri + Einstein Eqs:

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma_{ij}\sigma^{ij} - 8\pi G\, T_{ab}k^a k^v$$

$\lambda =$ affine parameter (null “distance” along each ray)

$\theta =$ the rate of expansion per unit area:

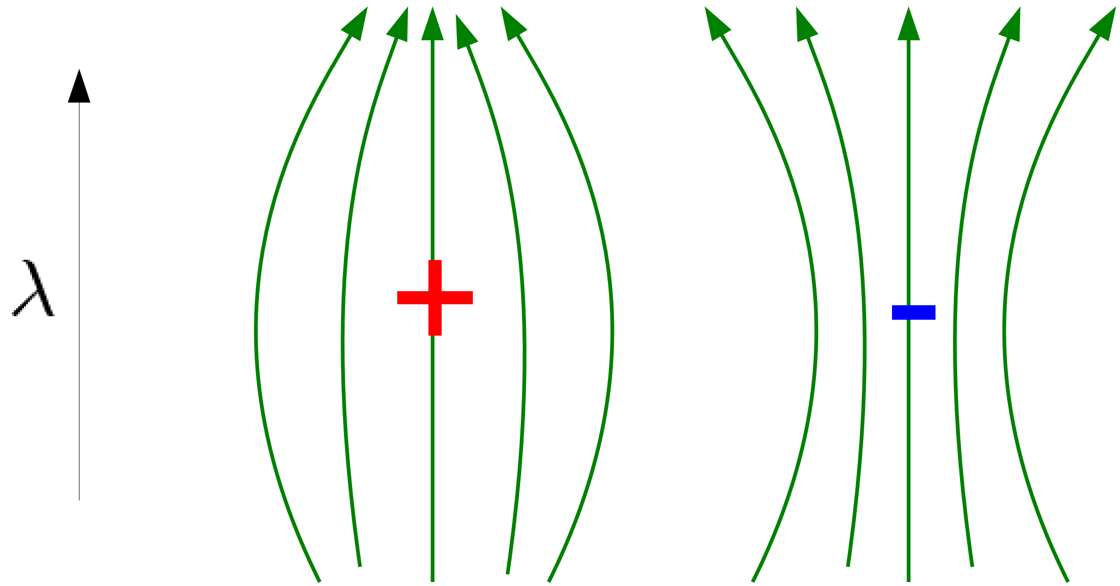
$\sigma_{ij} =$ rate of shearing into an ellipsoid

Assuming NEC, the right-hand side is negative, so if the surface is trapped, the lightrays must terminate at finite affine distance.

- They could terminate by crossing each other, but topologically they cannot **all** intersect each other unless space is finite (this step uses global hyperbolicity).

- otherwise, at least one of the lightrays must be inextendible (i.e. it hits a singularity).

All these geometric proofs from the null energy condition involve geometric focussing of lightrays!



Ignoring nonlinear terms, the Raychaudhuri equation relates the 2nd derivative of the Area *A* to the stress energy tensor:

$$\frac{d^2 A}{d\lambda^2} = -8\pi G T_{kk}$$

(k^a is "unit" null vector wrt λ)

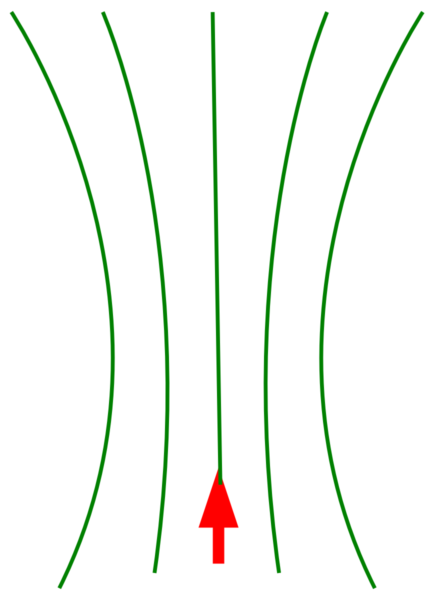
Quantum Energy Condition Violations

The Penrose theorem applies to *classical* general relativity. Can we extend it to quantum fields coupled to gravity?

In QFT, ***all*** local energy conditions can be violated in certain states although **-** energy must be balanced by **+** energy elsewhere:
(Klinkhamer 91, Folacci 92, Verch 00, various papers by Ford & Roman...)

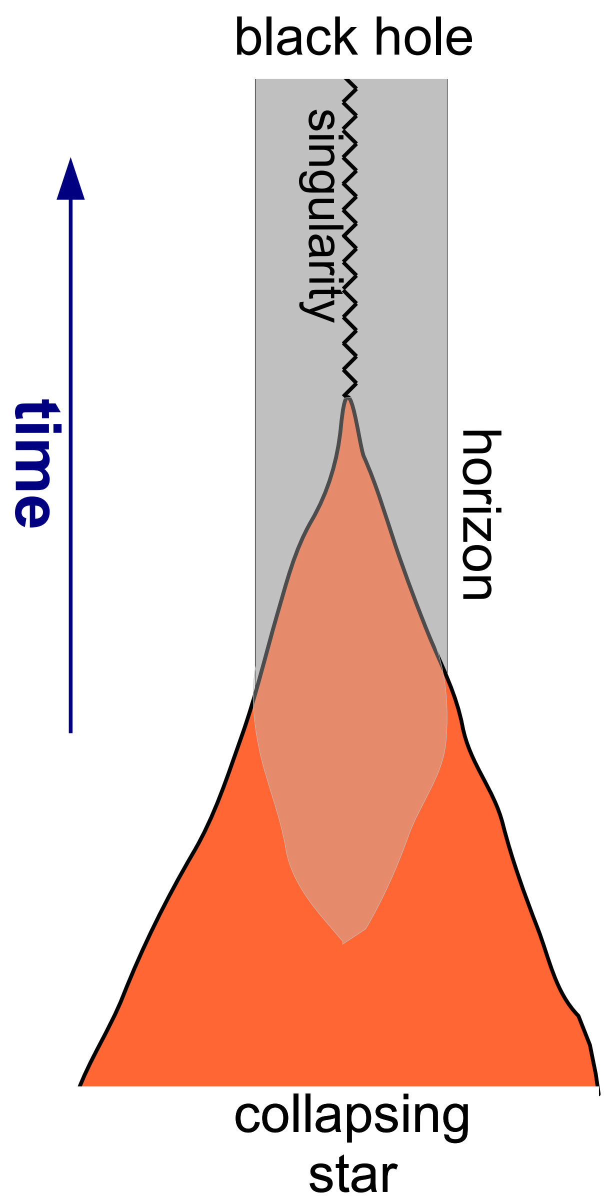
- Casimir effect (Brown-Maclay 69)
- moving mirrors (Davies-Fulling 76, 77)
- squeezed states (Braunstein, cf. Morris-Thorne 88)

…and more



So can all these global results be circumvented?

Hawking Area Increase Theorem

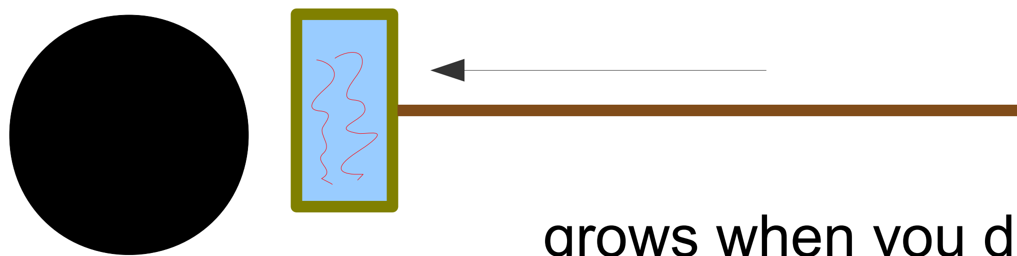


Hawking (71) proved that the total area of a black hole event horizon is always increasing (i.e. positive energy flux makes black holes grow)

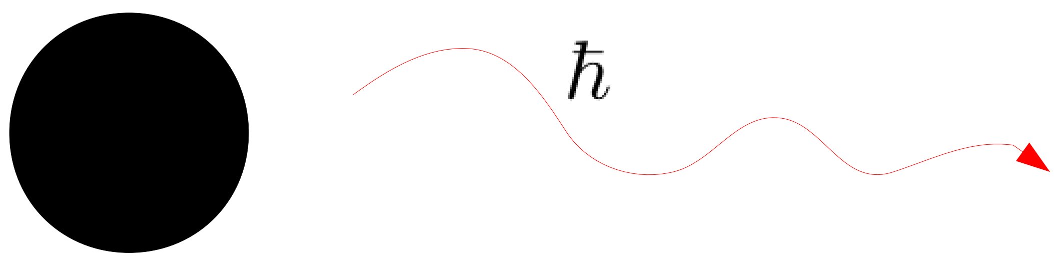
This also a classical result involving the null energy condition (the proof also involves focussing) but it *can* be generalized to quantum situations.

If *this* result has a quantum analogue, why not the singularity theorem & related results?

Black holes behave like thermodynamic systems



grows when you dump matter in



shrinks as Hawking radiation is emitted

black holes have temperature, and energy, thus an entropy

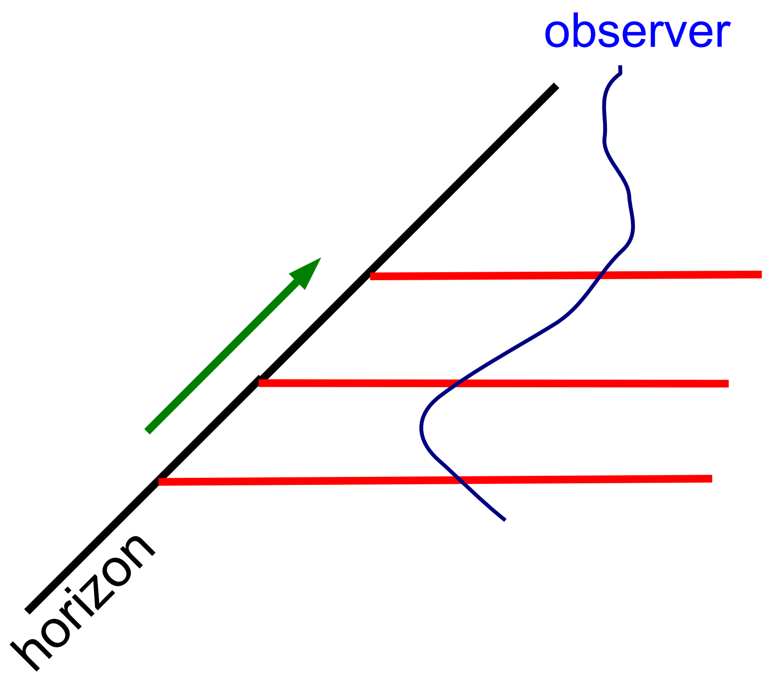
$$dE = TdS$$

proportional to the *area* of the horizon!

(also applies to other causal horizons e.g. de Sitter, Rindler)

Generalized Second Law

The outside of a causal horizon is an OPEN system—
info can leave (but not enter).



But the generalized entropy

$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{out}}$$

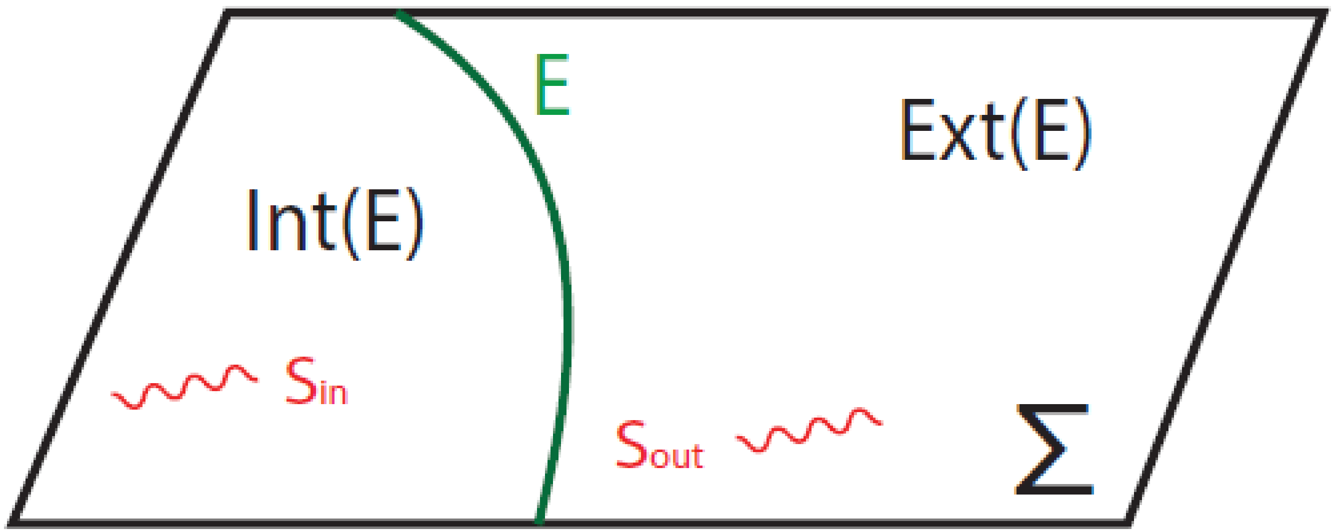
still increases. Area A of horizon contributes to entropy.

$$\frac{dS_{gen}}{dt} \geq 0$$

Generalized Second Law (GSL).

proved using lightfront quantization in arXiv:1105.3445 (AW)

Entanglement Entropy



Given any Cauchy surface Σ, and a surface E which divides it into two regions Int(E) and Ext(E), can define entanglement entropy:

$$S_{\rm ent} = -{\rm tr}(\rho \ln \rho)$$

where ρ is the density matrix restricted to one side or the other. for a pure total state, doesn't matter which side ($\rho_{\rm out}$ or $\rho_{\rm in}$), since $S_{\rm in} = S_{\rm out}$.

but for a mixed state, it does matter ($S_{\rm out} \neq S_{\rm in}$)

$S_{\rm ent}$ is UV divergent, but divergences are local.

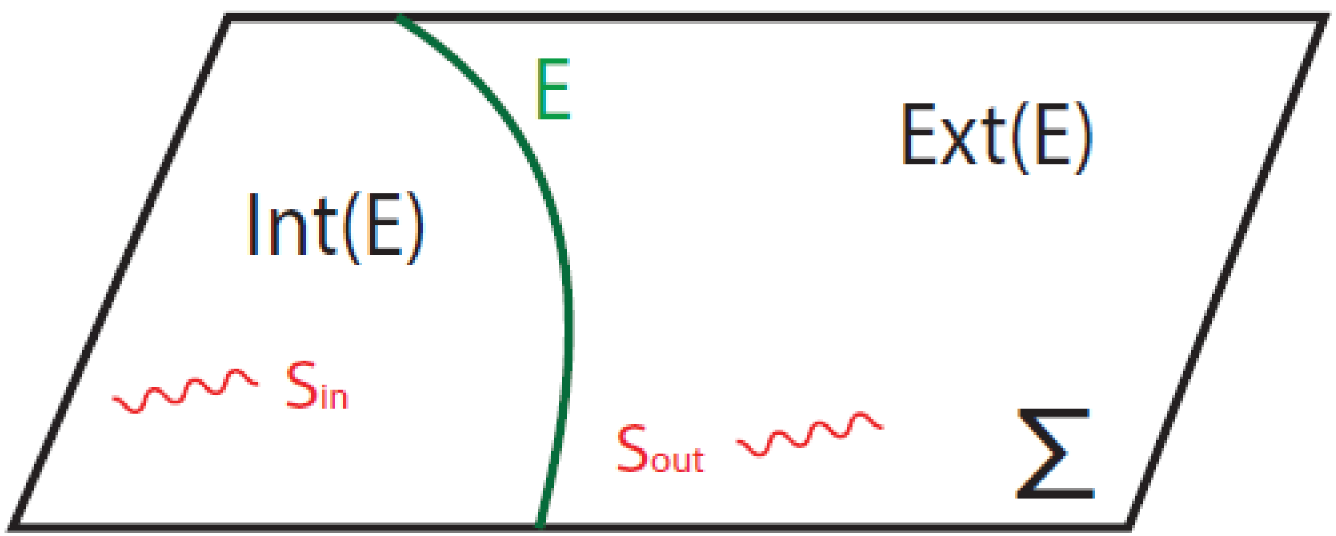
The Generalized Entropy

If the theory is GRAVITATIONAL, then we can also define a finite "generalized entropy" of E:

$$S_{\text{gen}} = \frac{\langle A \rangle}{4G\hbar} + S_{\text{out}} + \text{counterterms}$$

or we can use S_{in}.

counterterms are local geometrical quantities used to absorb EE divergences, (e.g. leading order area law divergence corrects 1/G)



hypothesis: related to gravitational state-counting somehow

Suggests way to extend classical GR proofs to “semiclassical” situations involving quantum fields...

just replace the area with the generalized entropy!

$$A \to 4G\hbar S_{\text{gen}}$$

Quantum Expansion

classical: area increase (per unit area) of :

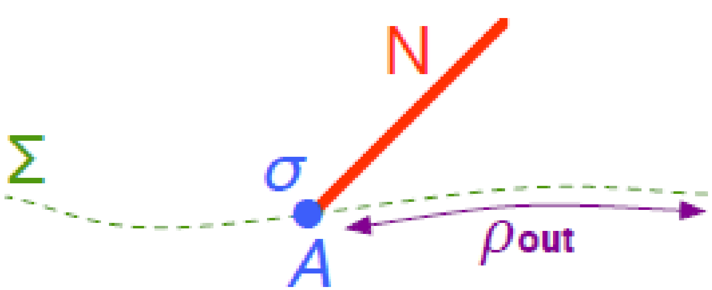
$$\theta = \lim_{A \to 0} \frac{1}{A} \frac{dA}{d\lambda}$$

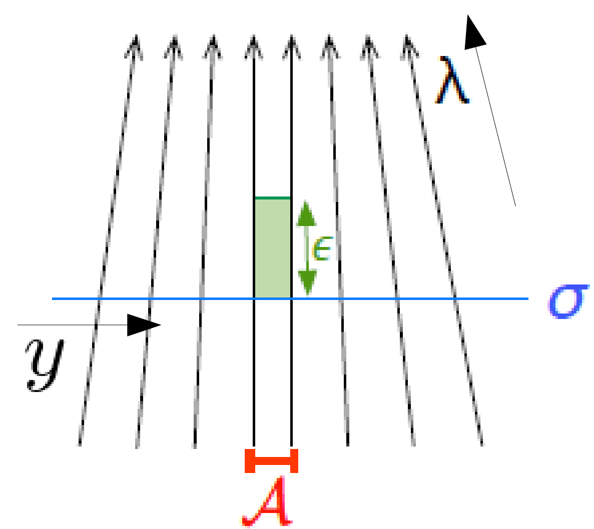
quantum: generalized entropy increase (still per unit area!)

$$\Theta = \lim_{A \to 0} \frac{4G\hbar}{A} \frac{dS_{\text{gen}}}{d\lambda} = \frac{4G\hbar}{a} \frac{\delta S_{\text{gen}}}{\delta \lambda(y)}$$

functional derivative of nonlocal quantity

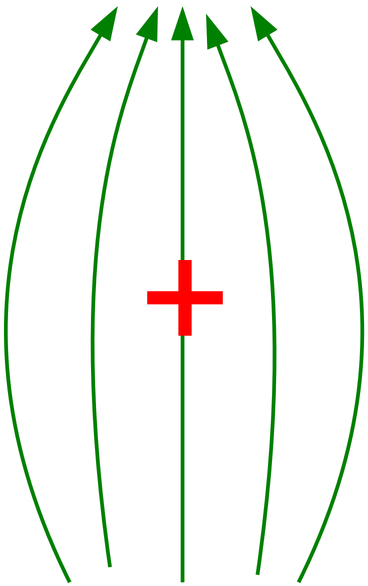
finite area element





Quantum Focussing Conjecture

asserts that the *second* functional derivative is negative:



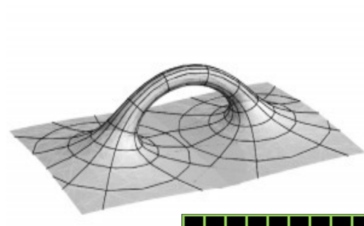
$$\frac{\delta}{\delta\lambda(y)}\Theta(y')\Big|_{\sigma} \leq 0$$

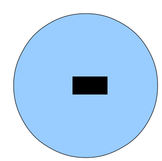
for *any* null surface, not just event horizons

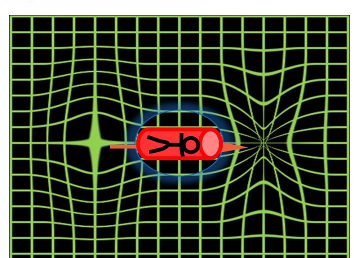
indicates that light rays always focus if you also include the entanglement entropy!

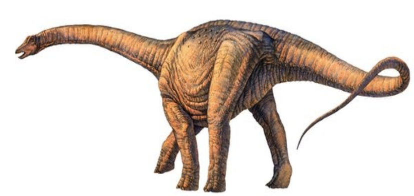
idea is to use this in place of null energy condition to prove similar results:

NO:

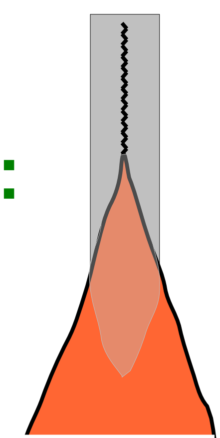








YES:



Different degrees of “quantum”-ness

Classical: general relativity, coupled to classical fields.

Semiclassical: QFT in curved spacetime,
plus infinitesimal backreaction on metric due to $T_{\mu\nu}$
(can also quantize linearized gravitons)

Perturbative: start taking into account graviton loops but remain at weak coupling

Full quantum gravity: (???)

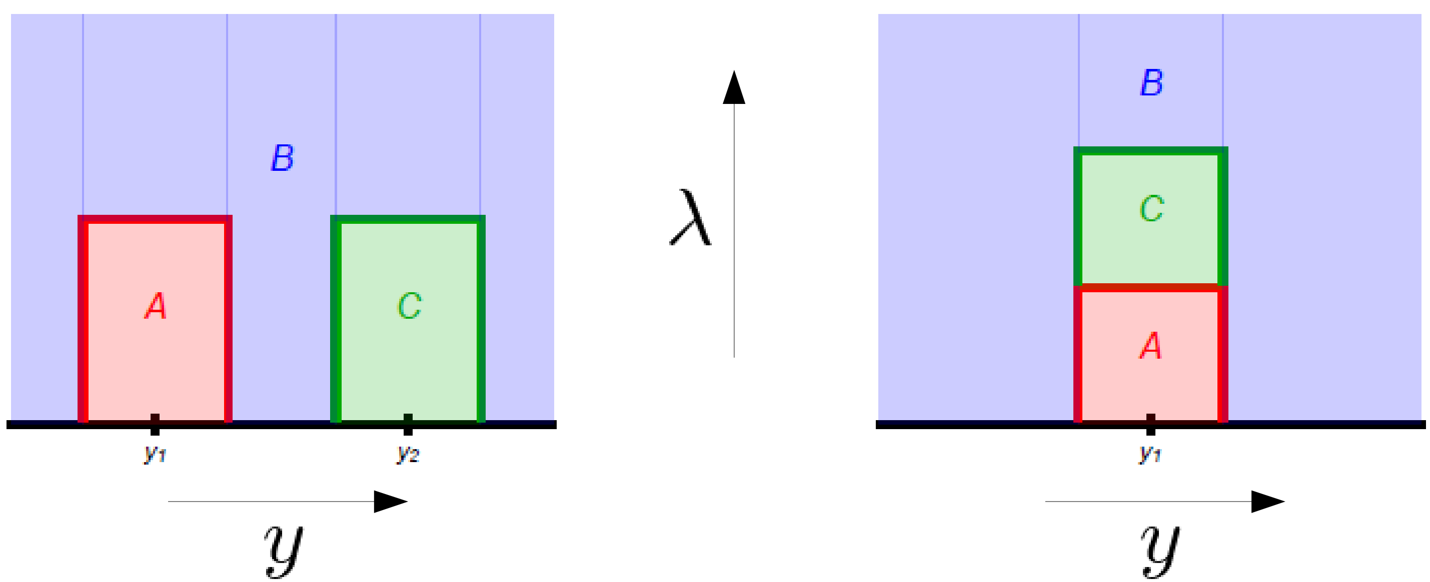
Towards a Proof of the Semiclassical QFC

$$\boxed{\frac{\delta}{\delta\lambda(y)}\Theta(y')|_\sigma \leq 0}$$

bilocal quantity: $f(y,y') + \delta(y-y')g(y)$

off-diagonal $y \neq y'$

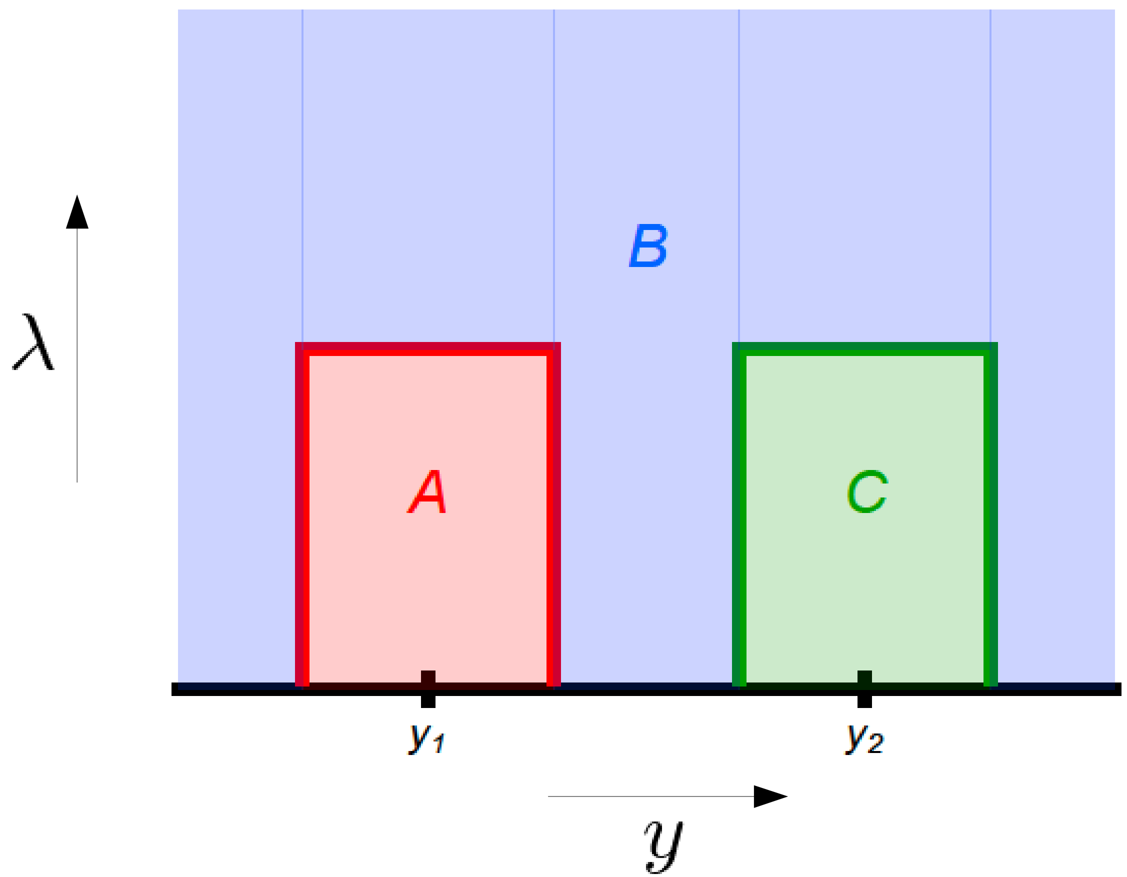
diagonal $y = y'$



Off-diagonal case, automatic for any quantum system

Strong Subadditivity:

$$S(AB) + S(BC) \geq S(ABC) + S(B)$$



Diagonal case, contact term requires special treatment

On nearly stationary null surface,
entropy of area & entropy can be same order in $G\hbar$

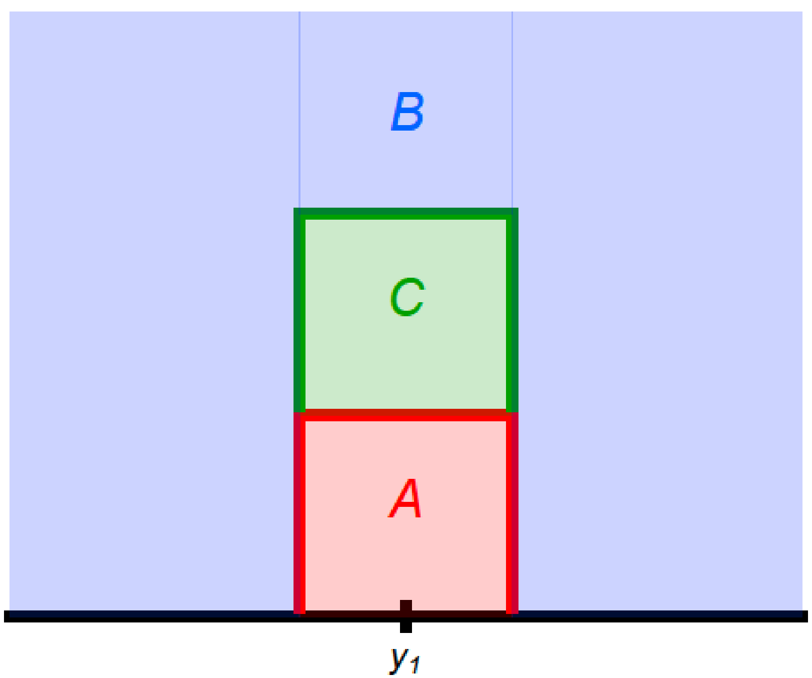
reduces to **Quantum Null Energy Condition:**

$$\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi A} S''_{\text{out}}$$

causes area focussing if you turn on gravity

$$\frac{d^2 A}{d\lambda^2} = -8\pi G T_{kk}$$

2nd derivative of entropy along region with small area A



Proofs of QNEC in QFT

- bosonic field theories with only relevant couplings

"Proof of the Quantum Null Energy Condition"
(Raphael Bousso, Zach Fisher, Jason Koeller,Stefan Leichenauer, AW)

Uses lightfront field theory, replica trick, + careful analytic continuations

- holographic field theories

"Holographic Proof of the Quantum Null Energy Condition"
(Jason Koeller, Stefan Leichenauer)

uses AdS/CFT duality (which relates a class of large N, strongly coupled gauge theories to GR in a higher dimensional spacetime).

but not yet proven in full generality...!

Higher Curvature Gravity

$$S_{\text{gen}} = \frac{\langle A \rangle}{4G\hbar} + S_{\text{out}} + \text{counterterms}$$

starting with a local correction to the GR action, e.g:

$$I = \int d^D x \sqrt{g}\, f(R_{abcd})$$

can derive entropy functional $\downarrow$ (in null coordinates v, u)

$$S = -\frac{2\pi}{\hbar} \int d^{D-2} x \sqrt{g} \left[\underbrace{4 \frac{\partial L_g}{\partial R_{uvuv}}}_{\text{Wald}} + \underbrace{16 \frac{\partial^2 L_g}{\partial R_{uiuj} \partial R_{vkvl}} K_{ij(u)} K_{kl(v)}}_{\text{Solodukhin, FPS, Dong, Miao...}} \right] + \mathcal{O}(K^4)$$

$$= \frac{A}{4G\hbar} \text{ for GR}$$

(extrinsic curvature corrections only matter for nonstationary null surfaces)

Higher Curvature Focussing Result

In any metric-scalar theory of gravitation w/ arbitrarily complex action

$$I = \int d^D x \sqrt{g}\, L(g^{ab}, R_{abcd}, \nabla R \ldots \phi, \nabla\phi \ldots) + I_{\mathrm{matter}}$$

for a linearized perturbation of g_{ab}, ϕ about a stationary null surface,

"A Second Law for Higher Curvature Gravity" (AW)

showed one can always construct an entropy density *s* that focusses:

$$T_{vv} = -\frac{2\pi}{\hbar}\frac{d^2 s}{dv^2}$$

the integral of this *s* agrees with "Dong entropy" for f(Riemann) actions!

Discussion

QFC is a novel spacetime thermodynamic principle:

- unifies geometry with information theory
- more local than black hole thermodynamics
- useful for extending GR theorems to quantum situations

In general it is a *conjecture* (hence the name), but in semiclassical situations where quantum effects are small, it can be reduced to the QNEC, a lower bound on T_{kk}. in terms of entanglement entropy.

- The QNEC can be proven for free, relevant & holographic cases, but more work is necessary to prove it for every decent QFT.

Quantum gravity can lead to flat spacetime field theory insights...!