Superstring Interactions
in a
Plane-wave Background

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Motivation

- A special case of AdS/CFT where the string theory is exactly solvable
  → we can go beyond supergravity
- Obtain exact formulas for certain non-protected field theory observables, e.g. the dimension of BMN operators $\Delta - J = \sqrt{1 + \frac{g_{YM}^2 N}{J^2} n^2}$

  [Berenstein, Maldacena, Nastase]
- Resurrect light cone string field theory.

  [Mandelstam, Cremmer & Gervais, Kaku & Kikkawa, Green & Schwarz, Brink]

BMN Double Scaling Limit

Consider $\mathcal{N}=4$ $SU(N)$ Yang-Mills, in a sector with $U(1)$ R-charge J. Then the $N \to \infty$ limit, with $\frac{J^2}{N}$, g^2_{YM}, and $\Delta - J$ fixed, is dual to IIB string theory in the pp-wave background.

(Note: $\lambda = g^2_{YM} N \to \infty$)

$$\boxed{\mu p^+ \alpha' = \frac{J}{\sqrt{\lambda}}, \quad \frac{2p^-}{\mu} = \Delta - J, \quad g^2_{YM} = 4\pi g_s}$$

[Berenstein, Maldacena, Nastase]

Effectively Perturbative Duality (A HOPE!

[GT = Constable, Freedman, Headrick, Minwalla, Motl, Postnikov, Skiba]

For finite but small

$\lambda' = \frac{\lambda}{J^2} = (\mu p^+ \alpha')^{-2}$ & $g_2 = J^2/N = \frac{4\pi g_s}{\lambda'}$

the string and field theories are both "perturbative." Of course, field theory calculations must be done at small λ and finite J, then $\lambda \to \infty, J \to \infty$

The miracles are that

- some quantities remain finite in this limit of field theory
- some of those agree with string calculations.

It is an interesting open problem to understand exactly which observables have this miraculous behavior, and to understand precisely the dictionary for making comparisons.

State-Operator Correspondence @ $g_2 = 0$

BMN identified a natural correspondence between certain 'BMN' operators in the gauge theory and the states of string theory on the pp-wave background.

$$\mathrm{Tr}[Z^J] \qquad\qquad |0\rangle$$

$$\mathrm{Tr}[\phi_i Z^J] \qquad\qquad (a_0^i)^\dagger |0\rangle \qquad i,j = 1,2,3,4$$

$$\mathrm{Tr}[D_i Z\, Z^J] \qquad\qquad (a_0^{i+4})^\dagger |0\rangle$$

$$\underbrace{\sum_m \mathrm{Tr}[\phi_i Z^m \phi_j Z^{J-m}]} \qquad\qquad (a_0^i)^\dagger (a_0^j)^\dagger |0\rangle$$

$$\sum (\quad) \exp[2\pi i m n/J] \qquad\qquad (a_n^i)^\dagger (a_{-n}^j)^\dagger |0\rangle$$

Here $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$, and ϕ_i are the six scalar fields in the gauge theory.

State - Operator Correspondence cont'd

By calculating the anomalous dimensions of BMN operators, one finds that

$$\underbrace{(\Delta - J)_m}_{} = \sqrt{1 + \frac{g_{YM}^2 N}{J^2} m^2}$$

contribution from any single impurity with phase m.

[BMN, Gross Mikhailov Roiban, Santambrogio Zanon]

String theory in the plane wave is exactly solvable [Metsaev]. The spectrum is

$$P^- = \frac{1}{2p^+\alpha'} \sum_{n=-\infty}^{\infty} N_n \sqrt{n^2 + (\mu p^+ \alpha')^2}$$

This establishes $\frac{2}{\mu} P^- = \Delta - J$

at the level of free string theory. ($g_2 = 0$)

Interactions?

A number of impressive papers have pushed the gauge theory calculations to higher order, with the hope of showing that $\frac{2}{\mu}p^- = \Delta - J$ continues to hold at the interacting level.

EVIDENCE SO FAR = $\emptyset$!!!

p^- acts on the Hilbert space of string field theory, and causes the splitting and joining of strings.

$\Delta - J$ acts on the set of BMN operators in the gauge theory, and mixes single- and multi-trace operators, e.g.

$$\langle :\mathcal{O}\mathcal{O}:(x) \ :\mathcal{O}:(0)\rangle \sim g_2\lambda' \log(x^2\Lambda^2)$$

How do we identify two operators acting on different spaces?

Need a basis... [BKPSS, GMR, CFHM]

Change of Basis II

The BMN proposal comes with a natural identification between string theory states and gauge theory operators, but it is clear that this dictionary must receive corrections.

$\langle$1-string state$|$2-string state$\rangle = 0$

$\langle$(single trace BMN operator)(double-trace)$\rangle = \mathcal{O}(g_2)$

In order to prove that the two operators $\frac{2}{\mu}p^-$ and $\Delta - J$ are equal, it is sufficient to show that they have the same eigenvalues.* However, we hope to do better: by guessing how to identify the string basis in the gauge theory, we hope to check $\frac{2}{\mu}p^- = \Delta - J$ at the level of matrix elements. (* = in progress)

Light-Cone String Field Theory

$\Phi[X(\sigma)]$ — string field

It is convenient to write it as

$$\Phi(x^-) = \int_0^\infty \frac{dp^+}{\sqrt{p^+}} \sum_{a \in F} \left[e^{ix^-p^+} |a\rangle A_a^\dagger(p^+) + e^{-ix^-p^+} |a\rangle A_a(p^+) \right]$$

It is simultaneously a state in F – the Fock space of a single string, and an operator in $\mathcal{H}$ – the Hilbert space of string field theory.

$A_a^\dagger(p^+)$ creates a string in state $|a\rangle$ with momentum p^+.

$$\Phi[X(\sigma)] = \langle X(\sigma) | \Phi \rangle$$

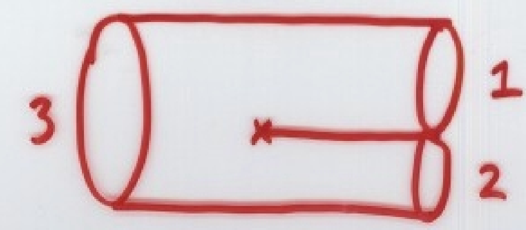
Cubic Interactions

The free Hamiltonian is

$$H_2 = i \int dx^- \, \mathcal{D}^8 X(\sigma) \, (\partial^+ \Phi) \, h(\Phi)$$

$$= \int dp^+ \sum_{a \in F} E_a \, A_a^\dagger(p^+) A_a(p^+)$$

The goal is to determine the cubic interaction



$$H_3 \sim \int \mathcal{D}X_1 \, \mathcal{D}X_2 \, \mathcal{D}X_3 \; \Phi(1) \, \Phi(2) \, \Phi(3) \; \delta(\Sigma p^+)$$

As in flat space, we can determine H_3 by requiring that the nonlinearly realized superalgebra closes to first order in the string coupling.

The expression as a path integral is only formal, and difficult to calculate (since the worldsheet theory is not conformal). Since we will want to compare specific matrix elements, it is more useful to represent H_3 as a state in the 3-string Hilbert space.

$$H_3 = \int dp_1^+ \, dp_2^+ \, dp_3^+ \; \delta(p_1^+ + p_2^+ + p_3^+)$$

$$\sum_{1,2,3 \in F} \Big(A_3^\dagger A_2 A_1 \underbrace{\langle 1 | \langle 2 | \langle 3 | H_3 \rangle}_{} + h.c. \Big)$$

This is just some function of p_1^+, p_2^+, p_3^+ which encodes the coupling of these 3 states.

Step I: The Measure

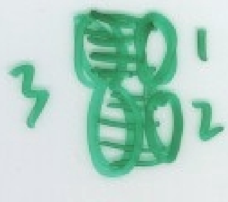
The measure in the path integral expresses the kinematic constraints

$$\Delta^8\left(X_1(\sigma) + X_2(\sigma) - X_3(\sigma)\right)$$

(momentum conservation) (and the fermionic partner of this...)

Construct the state $|V\rangle$ annihilated by

$$X_1(\sigma) + X_2(\sigma) - X_3(\sigma)$$
$$P_1(\sigma) + P_2(\sigma) + P_3(\sigma)$$



It has the form

$$|V\rangle = \exp\left[\frac{1}{2}\sum_{r,s=1,2,3}\sum_{m,n} a^{+}_{m(r)}\,\bar{N}^{(rs)}_{mn}\,a^{+}_{n(s)}\right]|0\rangle$$

The Neumann coefficients are very complicated functions of $(\mu p^+ \alpha')$...

The Necessity of a Prefactor

Consider a matrix element of $[Q^-, H]$ between on-shell states. If we had only $|H_3\rangle = |V\rangle$ then $[Q^-, H] = 0$ would require (at order g_s)

$$0 = \sum_{r=1}^{3} Q^-_{(r)} |V\rangle$$

$$= \sum \int d\sigma_{(r)} \left[p^I_{(r)} \gamma_I \bar{\theta}_{(r)} - i \partial_\sigma x^I_{(r)} \theta_{(r)} - i\mu x^I \gamma_I \Pi \bar{\theta}_{(r)} \right. |V\rangle$$

This is naively zero, but the operators acting on $|V\rangle$ are singular at the interaction point, giving a finite contribution to the integral.

$$p(\sigma)\, \bar{\theta}(\sigma) |V\rangle \sim \epsilon^{-1} |V\rangle$$

$$\partial_\sigma x(\sigma)\, \theta(\sigma) |V\rangle \sim \epsilon^{-1} |V\rangle$$

$$x(\sigma)\, \bar{\theta}(\sigma) |V\rangle \sim \epsilon^{0} |V\rangle$$

$(\sigma = \pi \alpha' p^+_{(1)} - \epsilon)$

Step II: The Prefactor

Supersymmetry requires the insertion of a local operator in the path integral.

Make an ansatz

$$|H_3\rangle = h_3(X, \tilde{X}, \Delta)|v\rangle$$

$$|Q_3^-\rangle = q_3^-(X, \tilde{X}, \Delta)|v\rangle$$

$$|\bar{Q}_3^-\rangle = \bar{q}_3^-(X, \tilde{X}, \Delta)|v\rangle$$

and require closure of the superalgebra. Here $X, \tilde{X}, \Delta$ are the unique operators which commute with all the kinematical symmetries. Remarkably, the functional forms of $h_3, q_3^-, \bar{q}_3^-$ are identical to flat space! Only the expressions for $X, \tilde{X}, \Delta$, when expressed in an appropriate basis, look different.

Finite μ $(\lambda' > 0)$ Neumann matrices

The expressions are implicit — they involve inverting $\infty \times \infty$ matrices.

[Recent progress: Schwarz] [more in progress...]

[For example, in flat space $\mu = 0$ we have

$$\bar{N}^{(12)}_{34} = \frac{1}{\sqrt{3}}(3y+1)(y-3)(1+y)(3-2y) \times (1-y)^{-2+3/y}\, y^{(1+3y)/(1-y)}$$
]

In most cases, even the first nontrivial correction for $\mu < \infty$ is not known!

We do know that fractional powers of λ' generically appear (for observables more complicated than the ones so far computed in the field theory).

Prototype: $\displaystyle\sum_{p=1}^{\infty} \frac{1}{p^2}\frac{1}{1+\lambda p^2} = \frac{\pi^2}{6} - \frac{\pi}{2}\sqrt{\lambda} + \mathcal{O}(\lambda)$!

Large μ (small λ') limit

We give 'expressions' for the matrix elements $\langle 1|\langle 2|\langle 3|H_3\rangle$ (and Q_3^- and $\bar{Q}_3^-$) for arbitrary states.

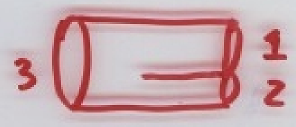
They are smooth functions of $\mu p^+\alpha'$ which interpolate smoothly between flat space $\mu=0$ and the highly curved limit $\mu=\infty$, $\lambda'=0$, in which the expressions are very simple, and agree with the (very limited) field theory calculations that have been done

$$\langle O_n^J|H_3|O_m^{J_1}\rangle|O^{J_2}\rangle$$

$$= \frac{4\mu g_s}{\pi}(1-y)\sin^2(\pi n y)$$

[0210102]
(First check)

$$y = J_1/J$$

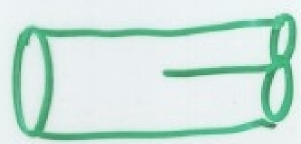


What is Σ?

Note for string field theory enthusiasts:

Σ is literally just V, the cubic vertex (without prefactor).

$$V : \mathcal{H}^{k} \longrightarrow \mathcal{H}^{k\pm 1}$$



So our proposal can be expressed as

(k-trace BMN operator)

$$\Updownarrow$$

$e^{g_2 V/2}$ | k-string state>

But so far we only know this for sure at order g_2. How do we know that? $\Rightarrow$

Moving along to order g_2^2 ...

This success only proves that we correctly identified the basis transformation between string states and BMN operators at g_2.
A basis-independent statement (e.g. eigenvalues) requires moving on to order g_2^2.
On the field theory side $(\Delta - J)$ has been diagonalized within the subspace of 2-impurity BMN operators:

$$(\Delta - J)_n = 2 + \lambda' \left[n^2 + \frac{g_2^2}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32\pi^2 n^2} \right) \right]$$

[Beisert, Kristjansen, Plefka, Semenoff, Staudacher; Constable, Freedman, Headrick, Minwalla]

Can we do the required one-loop SFT calculation?

Light-cone SUSic SFT at one loop

$$P^- = P^-_{(0)} + g_s P^-_{(1)} + g_s^2 P^-_{(2)} + \cdots$$

cubic interaction; single string splitting/joining

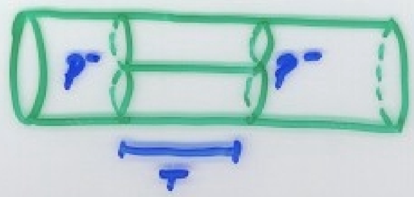
hard! but in principle determined by the SUSY algebra

$$Q = Q_{(0)} + g_s Q_{(1)} + \cdots$$

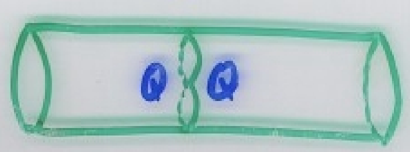
SUSY determines $P^-_{(2)} = \{Q_{(1)}, \bar{Q}_{(1)}\} + \cdots$

ignore for now

The eigenvalues therefore receive two contributions



$$\int_0^\infty dT \, \langle n| P^-_{(1)} e^{T(P^-_{(0)} - E_n)} P^-_{(1)} |n\rangle$$



$$\langle n| \{Q_{(1)}, \bar{Q}_{(1)}\} |n\rangle$$

Both of these terms are divergent, but their sum should be finite. The divergences come from high-energy intermediate states. We have performed a truncated calculation, where we include only intermediate states with two impurities. We believe this truncation is legitimate in the strict $\mu p^+ \alpha' \to \infty$ limit, but certainly a full calculation would be desirable.

[In the gauge theory, $\Delta - J$ has been diagonalized within the subspace of 2-impurity operators, not <u>all</u> BMN operators.]

We find perfect agreement!

(Modulo some factors of 2, in progress...)

CONCLUSIONS

- We have constructed light-cone SFT in a plane wave background for all $\mu p^+ \alpha'$; i.e. all λ'.
- The Neumann coefficients potentially encode a wealth of information about the gauge theory.
- We have used matrix elements of the light-cone SFT Hamiltonian $\langle 2\text{-string} | P_{(1)}^- | 1\text{-string} \rangle$ to identify the state-operator correspondence at order g_2.
- We have calculated matrix elements of the leading SFT contact term $\langle \{Q, \bar{Q}\} \rangle$ which will provide the first check of the BMN correspondence at order g_2^2.

Much work remains...