4/9/03 UNC-String Theory Seminar UNC-1

# Poisson sigma models, Batalin-Vilkovisky machinery & $L_\infty$-algebras

Warning! I am a mathematician trying to interpret work of several physicists. ~~The~~ Beyond my revisionist ~~his~~ presentation, there will be some of my joint work with Tom Lada and Ron Fulp of NCSU, the noncomm st. univ which was inspired by work of Behrens, Burgers and van Dam.

In their interpretation of Kontsevich's proof that any P manifold can be deformation quantized, C&F consider the following $\sigma$-model.

To provide a specific example of this correspondence and how it relates to the Batalin-Vilkovisky machinery, we turn to a Poisson sigma model of Cattaneo and Felder [3].

The fields of this Poisson $\sigma$-model are ordered pairs $(X, \eta)$ such that $X$ is a mapping from a 2-dimensional manifold $\Sigma$ into a Poisson manifold $M$ and $\eta$ is a section of the bundle $Hom(T\Sigma, X^*T^*M) \longrightarrow \Sigma$. These fields are subject to boundary conditions, namely they should satisfy the conditions: $X(u) = 0$ and $\eta(u)(v) = 0$ for arbitrary $u$ in the boundary of $\Sigma$ and for $v$ tangent to the boundary of $\Sigma$ at $u$. Observe that for each $u \in \Sigma$, we can regard $\eta(u)$ as a linear mapping from $T_u\Sigma$ into $T^*_{X(u)}M$. In local coordinates $\{u^\mu\}$ on $\Sigma$ and $\{x^i\}$ on $M$, we write $dX = (dX^j)\frac{\partial}{\partial x^j}$ and $\eta(\frac{\partial}{\partial u^\mu}) = \eta_{i,\mu}dx^i$. The Poisson structure is given by a *Poisson tensor* which is a skew-symmetric tensor on $M$

$$\alpha = \alpha^{ij}(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}) \tag{10}$$

which satisfies a Jacobi condition:

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0, \tag{11}$$

The action $S$ of the model is defined in such local coordinates by

$$S(X,\eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij} \circ X)(\eta_i \wedge \eta_j). \qquad (12)$$

The Euler Lagrange equations are:

$$E_{X^i} := d\eta_i + \tfrac{1}{2} \partial_i \alpha^{jk} (\eta_j \wedge \eta_k) = 0$$

$$E_{\eta_i} := -dX^i - \alpha^{ij} \eta_j = 0$$

The gauge symmetries of the action are parameterized by all sections $\beta$ of the bundle $X^*T^*M \longrightarrow \Sigma$ which vanish on the boundary of $\Sigma$. For each such $\beta$, define $\delta_\beta$ acting on the fields by

$$(\delta_\beta X)^i = (\alpha \circ X)(dx^i, \beta) \qquad (21)$$

and similarly for $(\delta_\beta \eta)_i$.
We may write $\delta_\beta$ as a variational symmetry:

$$\delta_\beta = \alpha^{ij}(X)\beta_j \frac{\partial}{\partial X^i} - (d\beta_i + (\partial_i \alpha^{jk})(X)\eta_j \beta_k)\frac{\partial}{\partial \eta_i}$$

Now Emma Noether had two variational theorems. The first relates symmetries to conserved quantities as is the better known. The second relates symmetries to relations among the EL eqns

In this example, the relations corresponding to $\delta_\beta$ are:

$$\alpha^{ik} E_{X^i} + d E_{\eta_k} - \partial_i \alpha^{jk} \eta_j E_{\eta_i} = 0$$

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## 5. First steps of the Batalin-Vilkovisky formalism

Rather than review the Batalin-Vilkovisky formalism in general as in [5, 2, 1], we illustrate it by example: the Poisson sigma model we have been considering. Batalin and Vilkovisky first construct a graded commutative algebra over ~~$Loc_E$~~ the algebra of local fcns with generators $X_i^+$ and $\eta^{+i}$, called 'anti-fields', $\gamma_i$ called 'ghosts' and $\gamma^{+i}$, called 'anti-ghosts. (~~If only the ghosts were used as generators, this would be a BRST algebra.~~)

By graded commutative, we mean polynomials in the even variables and exterior in the odd.

~~Function~~

These generators are bigraded, as indicated in the following table where the form degree is displayed as the top row and the ghost degree as the first column. *The graded commutativity is with respect to the sum of the ghost degree and the form degree (which we call the total degree).*

The assignments of degree (from left to right) and ghost number (from top to bottom) are given by

$$\begin{array}{cccc} & 0 & 1 & 2 \\ -2 & & & \gamma^{+i} \\ -1 & & \eta^{+i} & X_i^+ \\ 0 & X^i & \eta_i & \\ 1 & \gamma_i & & \end{array}$$

What's going on here? First let's look at the ghosts. If we adjoin just the ghosts, we can define a BRST operator aka ~~the~~ Chevalley Eilenberg differential:

$$\delta X^i = \alpha^{ij}(X)\gamma_j,$$
$$\delta \eta_i = -d\gamma_i - \partial_i \alpha^{jk}(X)\eta_j\gamma_k,$$
$$\delta \gamma_i = \frac{1}{2}\partial_i \alpha^{jk}(X)\gamma_j\gamma_k.$$

If $\alpha^{ij}(X) = \alpha^{ij}_k X^k$, we would be seeing a Lie algebra $\mathfrak{g}$ with basis $Z^i$ dual to the $\gamma_i$

What *is* a Lie algebra?

There are several current descriptions which Lie might not recognize! Mathematicians prefer a coordinate free version, but physicists seem to like bases. Let $\{Z^i\}$ be a basis for $\mathfrak{g}$ ~~with struc~~.
The algebra structure corresponds to a bracket in terms of structure constants

$$[Z^i, Z^j] = c^{ij}_k Z^k$$

satisfying the Jacobi identity

Similarly a representation of $\mathfrak{g}$ is a vector space $V$ w basis $\{Y^I\}$ and structure constants

$$Z^i Y^I = a^{iI}_J Y^J$$

satisfying an analog of Jacobi

Define $\{\gamma_i\}$ to be ~~a~~ dual ~~basis~~ Grassman variables to the $Z^i$

The above can be transcribed as

$$\delta \gamma_k = \tfrac{1}{2} c^{ij}_k \gamma_i \gamma_j$$

$$\delta Y^k = a^{ik}_J Y^J \gamma_i$$

The Jacobi & rep conditions hold iff $\delta^2 = 0$

$\delta$ is called a BRST operator or CE diff

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But in our $\sigma$-model, we have structure fcns not constants and so $\delta^2 \neq 0$ occurs but all is not lost. The great discovery of BV was how to work around this — by adding terms of higher order by introducing anti fields and antighosts. The antifields correspond to the shell, the $E_0$. Then we define another differential due to Kos, ~~means the anti-fields generate the Koszul complex with~~

$$
\begin{aligned}
d_{KT}\ X_i^+ &= d\eta_i + \frac{1}{2}\partial_i\alpha^{kl}(X)\eta_k \wedge \eta_l = E_{X^i} \\
d_{KT}\ \eta^{+i} &= -dX^i - \alpha^{ij}(X)\eta_j = E_{\eta_i}.
\end{aligned} \tag{31}
$$

Because of the Noether identities, the Koszul complex has non-trivial cohomology in ghost degree $-1$, namely the classes given by the formulas for the identities with $E_{X^i}$ and $E_{\eta_i}$ replaced by $X_i^+$ and $\eta^{+i}$ :

$$
-\alpha^{ij}(X)X_j^+ - \partial_k\alpha^{ij}(X)\eta_j \wedge \eta^{+k} - d\eta^{+i}. \tag{32}
$$

These classes can be killed by adjoining the anti-ghosts $\gamma^{+i}$ and defining

$$
d_{KT}\gamma^{+i} = -\alpha^{ij}(X)X_j^+ - \partial_k\alpha^{ij}(X)\eta_j \wedge \eta^{+k} - d\eta^{+i}. \tag{33}
$$

Thus the anti-ghosts occur precisely because of the identities identified by Noether. The second stage is due to Tate.

The pairing between symmetries and identities is now expressed as the pairing between ghosts and anti-ghosts, which plays a crucial role in the Batalin-Vilkovisky anti-bracket, but first the anti-fields and anti-ghosts are themselves subject to symmetries corresponding to $\delta_\beta$ as follows:

$$
\begin{aligned}
\delta X_i^+ &= \partial_i\alpha^{kj}(X)X_k^+\gamma_j \\
\delta\eta^{+i} &= \partial_k\alpha^{ij}(X)\eta^{+k}\gamma_j \\
\delta\gamma^{+i} &= \partial_k\alpha^{ij}(X)\gamma^{+k}\gamma_j.
\end{aligned} \tag{34}
$$

## 6. The Batalin-Vilkovisky anti-bracket and total differential

The hoped for total differential $D$ will be obtained by adding 'terms of higher order' to $d_{KT} + \delta$, which does not square to zero. To do this in general, Batalin and Vilkovisky introduce an 'anti-bracket' ( , ) which is defined in terms of distributional derivatives of functionals of the fields and anti-fields.

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The pairing defines the antibracket on generators:

$$(X^i, X_j^+) = \delta_j^i \qquad (35)$$
$$(\eta_j, \eta_+^i) = \delta_j^i$$
$$(\gamma_j, \gamma_+^i) = \delta_j^i.$$

~~and is extended as a bidern~~

The BV anti-bracket extends this as a graded biderivation with respect to ghost degree and in this example can be written as $(A, B) =$

slide?

$$\sum_\alpha \int_\Sigma (-1)^{|\phi_\alpha|(|\phi_\alpha|+|A|)} \left( \frac{\partial A}{\partial \phi^\alpha} \wedge \frac{\partial B}{\partial \phi_\alpha^+} - (-1)^{(deg(\phi_\alpha)+|A|+1)} \frac{\partial A}{\partial \phi_\alpha^+} \wedge \frac{\partial B}{\partial \phi^\alpha} \right) \qquad (36)$$

where $|C| = gh(C) + deg(C)$ denotes the Grassman parity of $C$ ($C$ is either a field or a function of fields). Note that physicists prefer to use both left and right derivatives and hence exhibit a different set of signs.

The antibracket obeys the graded commutativity relation

$$(A, B) = -(-1)^{(\mathrm{gh(A)}-1)(\mathrm{gh(B)}-1)}(B, A)$$

and the Leibniz rule

$$(A, BC) = (A, B)C + (-1)^{(\mathrm{gh(A)}-1)\mathrm{gh(B)}} B(A, C), \qquad (37)$$

which emphasizes the resemblance to a Poisson bracket. The only difference from a graded Poisson bracket is that the bracket shifts the degree by 1 and the several identities (skew-commutativity, Jacobi and Leibniz) inherit certain signs. Such an 'odd' Poisson bracket is also known as a **Gerstenhaber bracket** [4].

Now it is possible to express $d_{KT} + \delta$ in the form $(S^0 + S^1, \ )$ where

$$S^0 = (X, \eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij} \circ X)(\eta_i \wedge \eta_j),$$

our original action, and $S^1$ is

$$\int_\Sigma X_i^+ \alpha^{ij}(X)\gamma_j - \eta^{+i} \wedge (d\gamma_i + \partial_i \alpha^{kj}(X)\eta_k \gamma_j) - \frac{1}{2}\gamma^{+i}\partial_i \alpha^{jk}(X)\gamma_j \gamma_k. \quad (38)$$

which incorporates the gauge symms & the EoM cohomologically

Corresponding to the fact that $(d_{KT} + \delta)^2 \neq 0$, we have

$$(S^0 + S^1, S^0 + S^1) \neq 0.$$

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The additional terms in the differential $D$ we seek will be found by extending $S^0 + S^1$ by terms of higher order to achieve the full BV action $S_{BV}$.

Batalin and Vilkovisky show that, in much more general situations, one can add terms $S^i$ of ghost degree $i > 1$ to achieve a total $S_{BV}$ such that

$$(S_{BV}, S_{BV}) = 0.$$

The reason for this is that the $d_{KT}$ homology vanishes in appropriate degrees.

In the Cattaneo-Felder model, only one more term is needed:

$$S^2 = \int_\Sigma -\frac{1}{4}\eta^{+i} \wedge \eta^{+j} \partial_i \partial_j \alpha^{kl}(X)\gamma_k \gamma_l. \tag{43}$$

Thus the total Batalin-Vilkovisky generator is

$$\begin{aligned} S_{BV} \;=\; & \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i \wedge \eta_j \\ & + X_i^+ \alpha^{ij}(X)\gamma_j - \eta^{+i} \wedge (d\gamma_i + \partial_i \alpha^{kl}(X)\eta_k \gamma_l) - \frac{1}{2}\gamma_+^i \partial_i \alpha^{jk}(X)\gamma_j \gamma_k \\ & -\frac{1}{4}\eta^{+i} \wedge \eta^{+j} \partial_i \partial_j \alpha^{kl}(X)\gamma_k \gamma_l. \end{aligned} \tag{44}$$

We can then work out ~~an ext~~ the formulas for $D$ applied to the 6 kinds of generators and tease out the formulas for $\delta$, $d_{KT}$ and more.

But what does more ~~signify~~ and how does ~~this relate~~ to more classic examples of gauge symmetries without field dependence ~~as~~ in BBvD?

Here's an ~~better~~ alternative that looks better.

C&F ~

Recall that in total degree 0 we have
$X^i, \eta^{+i}, \gamma^{+i}$
and in total degree 1
~~$\eta_i$~~, $\gamma_i, \eta_i$ and $X_i^+$

Cattaneo and Felder thought to take the corresponding sums

$$\tilde{X}^i = X^i + \eta^{+i} + \gamma^{+i}$$
$$\tilde{\gamma}_i = \gamma_i + \eta_i + X_i^+$$

The total $\tilde{\delta}$ corresponding to $S_{BV}$ now looks like

$$\tilde{\delta}\tilde{X}^i = DX^i + \alpha^{ij}(\tilde{X})\tilde{\gamma}_j$$
$$\tilde{\delta}\tilde{\gamma}_i = D\tilde{\eta}_i + \tfrac{1}{2}\partial_i\alpha^{jk}(\tilde{X})\tilde{\gamma}_j\tilde{\gamma}_k$$

where $D$ is the de Rham differential on $\Sigma$, but now $\tilde{\delta}^2 = 0$.

And now we see exactly in disguise the formalism of BBvD for handling field dependent gauge symmetries. They would write

~~$\delta_{\tilde{Z}}\tilde{X}^i = T($~~

$$\delta_{\tilde{Z}}\tilde{X} = T(\tilde{X}, \tilde{Z}) = \sum T_n(\tilde{X}, \dots, \tilde{X}, \tilde{Z})$$

$$\text{and } [\delta_{\tilde{Z}_1}, \delta_{\tilde{Z}_2}]\tilde{X} = C(\tilde{X}, \tilde{Z}_1, \tilde{Z}_2)$$
$$= \sum C_n(\tilde{X}, \dots, \tilde{X}, \tilde{Z}_1, \tilde{Z}_2)$$

Their requirements on the relations among $T \otimes C$ ~~ab~~ corr UNC-9
precisely to $\delta^2 = 0$.

Here's ~~a final~~ another way to look at what's going on.

There's a notion of graded Lie algebra which means the underlying vector space ~~#~~ is a sum of pieces $\mathfrak{g} = \bigoplus_n \mathfrak{g}_n$ or

if you prefer super Lie algebras have $\mathfrak{g} = \mathfrak{g}_{odd} \oplus \mathfrak{g}_{even}$. In either case

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

s/t $\mathfrak{g}_p \otimes \mathfrak{g}_q \longrightarrow \mathfrak{g}_{p+q}$ and is graded skew and satisfies a version of Jacobi with some signs

~~This data packs up nicely if we consider $\Lambda \mathfrak{g}$, that is regrade by 1 and take the graded symmetric algebra. Then extend the bracket as a coderivation $\delta$~~

~~If $\mathfrak{g}$ has a differential~~

~~$\delta^2 = 0$ encodes Jacobi~~

~~If $\mathfrak{g}$ has a differential of its own~~

RU 3

**Definition 1.3 (Homotopy Lie algebras).** A strong *homotopy Lie algebra* is a complex $V = \sum_{i\in\mathbb{Z}} V_i$ with a differential $Q$, $Q^2 = 0$, of degree 1 and a collection of $n$-ary brackets:

$$[v_1, \ldots, v_n] \in V, \qquad v_1, \ldots, v_n \in V,\ n \geq 2,$$

which are homogeneous of degree $3 - 2n$ and super (or graded) symmetric:

$$[v_1, \ldots, v_i, v_{i+1}, \ldots, v_n] = (-1)^{|v_i||v_{i+1}|}[v_1, \ldots, v_{i+1}, v_i, \ldots, v_n],$$

$|v|$ denoting the degree of $v \in V$, and satisfy the relations

$$(1.3)\quad Q[v_1, \ldots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \ldots, Qv_i, \ldots, v_n] = \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \ \sum_{\substack{\text{unshuffles } \sigma:\\ \{1,2,\ldots,n\}=I_1\cup I_2,\\ I_1=\{i_1,\ldots,i_k\},\ I_2=\{j_1,\ldots,j_{l-1}\}}} \epsilon(\sigma)[[v_{i_1}, \ldots, v_{i_k}], v_{j_1}, \ldots, v_{j_{l-1}}],$$

where $\epsilon(i) = (-1)^{|v_1|+\cdots+|v_{i-1}|}$ is the sign picked up by taking $Q$ through $v_1, \ldots, v_{i-1}$, $\epsilon(\sigma)$ is the sign picked up by the elements $v_i$ passing through the $v_j$'s during the unshuffle of $v_1, \ldots, v_n$, as usual in superalgebra.

which is graded skew $[X,Y] = -(-1)^{\cdot\cdot}[Y,X]$ R~~H~~

where $(-1)^{XY}$ means $(-1)^{\deg X \deg Y}$

and satisfies Jac w signs, easiest to remember, Deform

The CE complex needs only minor modification

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Define $\uparrow\mathfrak{g} = s\,\mathfrak{g} = \mathfrak{g}[\overset{?}{\pm}1]$ by

$$(\uparrow\mathfrak{g})_{n+1} = \mathfrak{g}_n$$

~~Let A~~ For a graded vector space $V$ let $\Lambda V$ = graded symm alg gen by $V$

$$\simeq E(V_{odd}) \otimes S(V_{even})$$

Define a coderivation $\delta$ on $\Lambda\uparrow\mathfrak{g}$

by $\delta(\uparrow X) = -\uparrow dX$

$\delta(\uparrow X \wedge \uparrow Y) = \uparrow[X,Y]$

and extend as a coder

i.e.

$$\delta(\uparrow X_1 \wedge \cdots \uparrow X_n) = \sum \pm \quad \wedge \uparrow dX_i$$
$$+ \sum \pm \uparrow[X_i, Y_j] \wedge \cdots$$

$\delta^2 = 0$

Expressed that way, one could long ago have asked - what if we add terms $\delta(\uparrow X_1 \wedge \cdots \wedge \uparrow X_n) = \uparrow[X_1, \ldots, X_n]$ & still asked for $\delta^2 = 0$

Now historically that's not what happened but

UNC That's what's hiding in BV

RU only { Rather the associative analog arose in ~~topology~~ in terms of the chains on a based loopspace and then in the def theory of rational homotopy types.

Let's cut to the chase and analyze the pieces of such a $\delta$

Call the pieces $\delta_n$ which correspond ~~If $\delta^2=0$ then~~ to $[\ ,\ \dots,\ ]_n$ with $\delta_1 \Leftrightarrow d$

$$\delta^2=0 \Rightarrow \delta_1^2=0 \Rightarrow d^2=0$$

$$\Rightarrow \delta_1\delta_2 \mp \delta_2\delta_1 = 0$$

i.e. $\delta_2$ is still a chain map $\equiv \delta_1$ is a derivation of $[\ ,\ ]$

but something new next

Instead of Jac $\Leftrightarrow \delta_2\delta_2=0$

we now have $\delta_2\delta_2 = \delta_1\delta_3 + \delta_3\delta_1$

so $\quad \mathrm{Jac}(X,Y,Z) = d[X,Y,Z] + [dX,Y,Z] + \cdots$

Jacobi ~~fail~~ of ~~a~~ closed elements fails by an "exact term

or

"Jacobi holds up to $\simeq$

Notice that the $L_\infty$ algebra is $L = L_0 \oplus L_1$
where ~~$Z^i \in L_0$ and $X^i_j \eta_i$~~
$\tilde{X} \in L_1$ and $\tilde{\eta} \in L_0$
but $L_0$ is <u>not</u> a subalgebra.

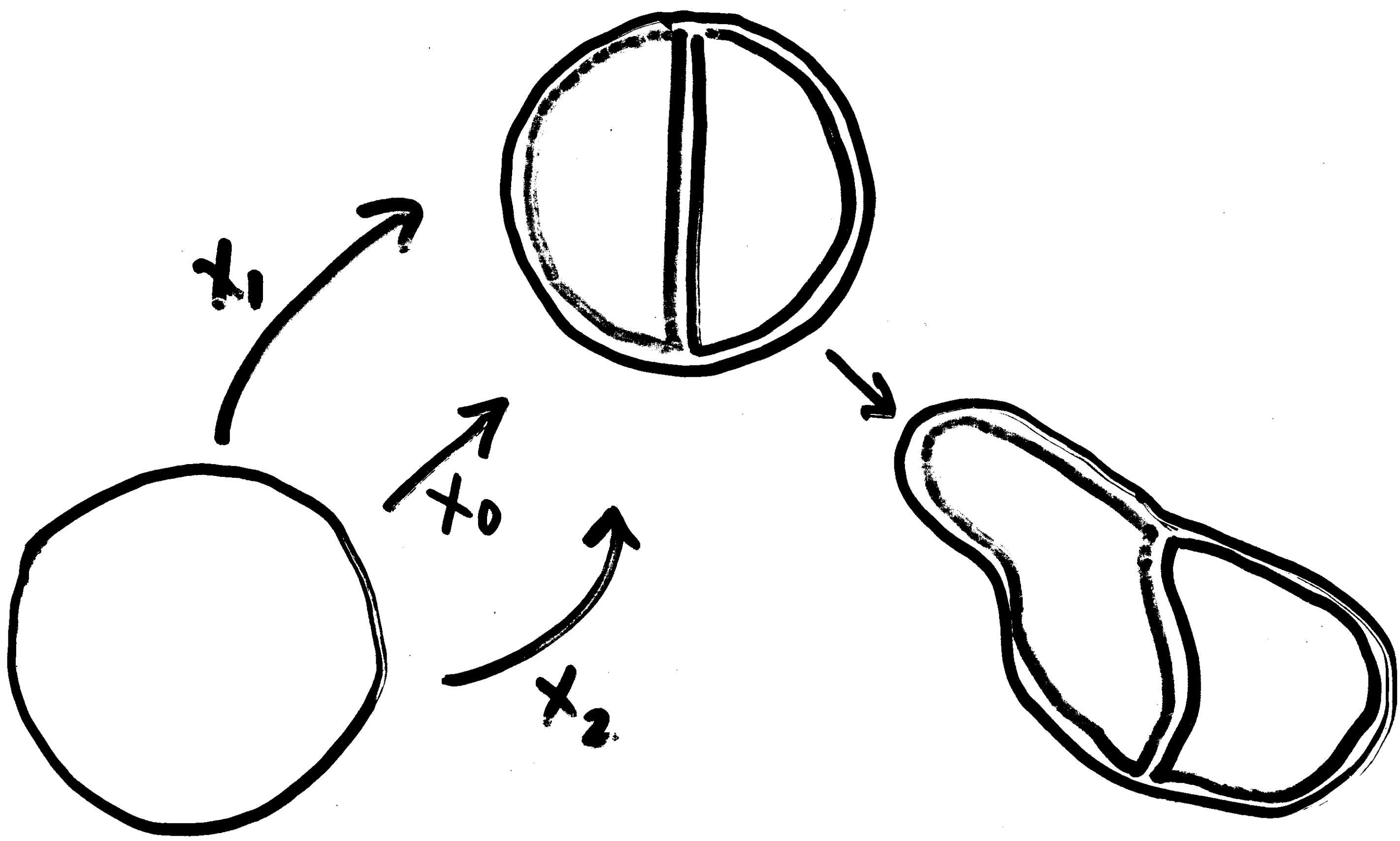
What is the physical significance of the individual terms?

First write $S_{BV} = \int \tilde{\eta}_i d\tilde{X}^i \pm \alpha^{ij}(\tilde{X}) \tilde{\eta}_i \tilde{\eta}_j$

then expand $\alpha^{ij}(\tilde{X})$ to see $n$-point functions for field interactions. This works wherever BV applies, e.g. to all the BBvD examples.

Another very important ~~case of an~~ ~~Locally~~ physically example of an $L_\infty$-structure is in Zwiebach's CSFT where the fields are functions on the space of closed strings and the convolution product of such fns corresponds to closed string decomposition

RU pix




$$[\varphi, \psi](X_0) = \iint \varphi(X_1)\, \psi(X_2)\, dX$$

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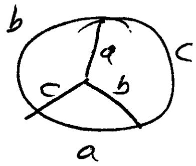
Here the higher order $L_\infty$ brackets correspond to n-point functions determined by further decompositions e.g. 

where only if a or b or c $= \pi$ is the result an iterated [Σ, T, J].

~~Another pl~~

Quite a different place where $L_\infty$-algebras appear naturally is in def theory as in my work w Mike Schlessinger and in K's def q which brings us full circle to the beginning of this talk.