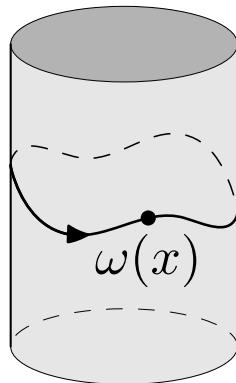
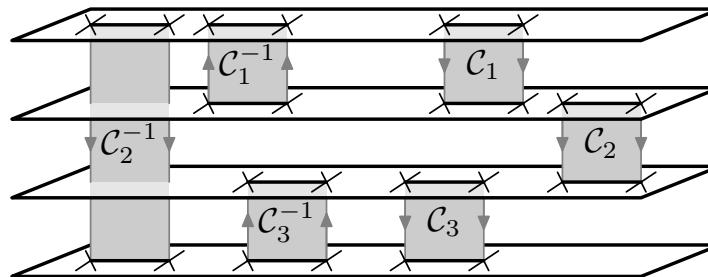
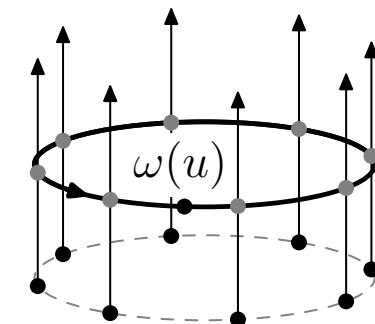


Strings on $AdS_5 \times S^5$, $\mathcal{N} = 4$ Super Yang Mills and Algebraic Curves



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Based on work with V. Kazakov, K. Sakai, K. Zarembo: hep-th/0410253.

Large Spin Limits of AdS/CFT

AdS/CFT: String energies & gauge dimensions match: $\{E\} = \{D\}$

Proposal: Consider states with large spin J on S^5 /of flavour $\mathfrak{so}(6)$

- BMN limit; non-planar and near $\mathcal{O}(1/J)$ extensions.
- Semiclassical spinning strings:

$\begin{bmatrix} \text{Berenstein} \\ \text{Maldacena} \\ \text{Nastase} \end{bmatrix} \dots$
 $\begin{bmatrix} \text{Frolov} \\ \text{Tseytlin} \end{bmatrix}$

$\mathcal{O} \sim \text{Tr } \phi_1 \dots \phi_1 \phi_2 \dots \phi_2 \phi_1 \dots \phi_1 \phi_2 \dots \phi_2 \longleftrightarrow \text{long classical strings.}$

Effective coupling constant $\lambda' = \frac{\lambda}{J^2}.$

- String theory: Expansion in λ' and $1/J \sim 1/\sqrt{\lambda}$,
- Gauge theory: ℓ -loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in λ' apparently equivalent to expansion in λ . Compare!

Three-loop mismatch in near BMN limit.

Similar disagreement for spinning strings.

$\begin{bmatrix} \text{Callan, Lee, McLoughlin} \\ \text{Schwarz, Swanson, Wu} \end{bmatrix} \begin{bmatrix} \text{Callan} \\ \text{McLoughlin} \\ \text{Swanson} \end{bmatrix}$
 $\begin{bmatrix} \text{Serban} \\ \text{Staudacher} \end{bmatrix}$

Outline

- String Solutions:
What has been done and what cannot be done?
- Classical Bosonic Strings on $AdS_5 \times S^5$:
Construction of an algebraic curve from a solution.
Classification of solutions.
- $\mathcal{N} = 4$ Gauge Theory in the Thermodynamic Limit:
Construction of an algebraic curve for a state.
- Comparison of curves.

General Assumptions:

- No string interactions, planar.
- Classical bosonic strings.
- One-loop gauge theory.

Strings in Flat Space

Equations of motion: $\partial_+ \partial_- \vec{X} = 0$.

Solved by Fourier transformation. Mode decomposition:

$$\vec{X}(\tau, \sigma) = \vec{x}_0 + \tau \vec{p} + \operatorname{Re} \sum_{n \neq 0} \vec{a}_n \exp(i|n|\tau + in\sigma)$$

subject to Virasoro constraint $(\partial_\pm \vec{X})^2 = 0$.

- ★ Solutions classified by $|\vec{p}|$ and amplitudes $|\vec{a}_n|$.
- ★ Quantize string modes $\vec{a}_n \rightarrow \vec{\alpha}_n$ (as well as \vec{x}_0, \vec{p}).

String Hamiltonian:

$$H = \frac{1}{\alpha'} \sum_n |n| \vec{\alpha}_n^\dagger \vec{\alpha}_n.$$

Strings in (Near) Plane Waves

Consider $AdS_5 \times S^5$ as expansion around plane waves.

Solution for strings on plane waves

[<sub>Metsaev
hep-th/0112044</sub>] [<sub>Metsaev
Tseytlin</sub>]

$$\vec{X}(\tau, \sigma) = \text{Re} \sum_n \vec{a}_n \exp(i\omega_n \tau + in\sigma), \quad \omega_n = \sqrt{1/\lambda' + n^2}.$$

- ★ Solutions classified by amplitudes $|\vec{a}_n|$.
- ★ Quantize string modes $\vec{a}_n \rightarrow \vec{\alpha}_n$.

Recover $AdS_5 \times S^5$ Hamiltonian as expansion in $1/R$

[<sub>Callan, Lee, McLoughlin
Schwarz, Swanson, Wu</sub>]

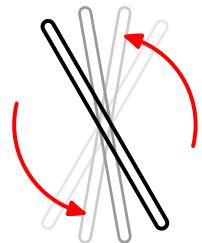
$$H = \sqrt{\lambda'} \sum_n \omega_n \vec{\alpha}_n^\dagger \vec{\alpha}_n + \mathcal{O}(\vec{\alpha}^4/R) + \mathcal{O}(\vec{\alpha}^6/R^2) + \dots$$

- Coupling of arbitrarily many modes.
- $\mathcal{O}(1/R^2)$ already very complicated.

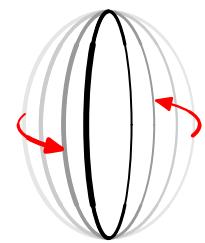
Spinning Strings

Many examples investigated:

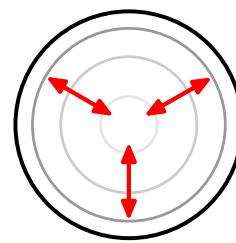
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hep-th/0209047] [Frolov
Tseytlin] ...



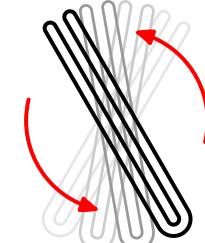
folded



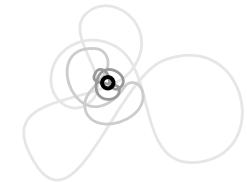
circular



pulsating



higher modes



plane waves

Ansatz, e.g. string on $\mathbb{R}_t \times S^2$: Energy $\mathcal{E} = E/\sqrt{\lambda}$, spin $\mathcal{J} = J/\sqrt{\lambda}$.

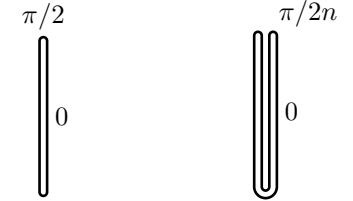
$$t(\tau, \sigma) = \mathcal{E} \tau, \quad \vec{X}(\tau, \sigma) = \begin{pmatrix} \sin \vartheta(\sigma) \cos \mathcal{J} \tau \\ \sin \vartheta(\sigma) \sin \mathcal{J} \tau \\ \cos \vartheta(\sigma) \end{pmatrix}.$$

Solve equations of motion and Virasoro constraint

$$\vartheta(\sigma) = \text{am}(\mathcal{E}(\sigma - \sigma_0), \eta), \quad \mathcal{J} = \eta \mathcal{E}.$$

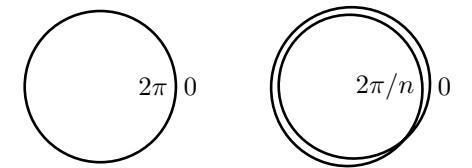
Global Charges

Folded string: $\vartheta(0) = 0$ and $\vartheta'(\pi/2n) = 0$



$$J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n}{\pi} K(1/\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\eta\pi} K(1/\eta).$$

Circular string: $\vartheta(0) = 0$ and $\vartheta(2\pi/n) = 2\pi$



$$J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n\eta}{\pi} K(\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\pi} K(\eta).$$

Global charges of generic solutions

$$J_k = \sqrt{\lambda} \mathcal{J}_k(\eta_a), \quad S_k = \sqrt{\lambda} \mathcal{S}_k(\eta_a), \quad E = \sqrt{\lambda} \mathcal{E}(\eta_a)$$

with algebraic, elliptic, hyperelliptic, ... functions of moduli $\{\eta_a\}$.

- Why elliptic functions? What is the meaning of moduli?

Strings on $AdS_5 \times S^5$

- Too difficult to solve the equations of motion in general.
No direct way to quantization as in flat space or plane waves.
- Very difficult to expand around plane waves.
Only expansion ...

Now what?

- Give up on finding exact energy spectrum.
- Classify solutions to understand structure of spectrum.
- Try to quantize that. (...?)

How?!

- Extract all conserved charges: Lax pair, monodromy.
- Investigate their analyticity properties.
- Reconstruct the corresponding algebraic curve.
- Discretize the curve. (Later!)

Bosonic String on S^5 and AdS_5

Embedding of S^5 in \mathbb{R}^6 . Coordinates $\vec{X}(\tau, \sigma) \in \mathbb{R}^6$ with $\vec{X}^2 = 1$.
Standard sigma model action

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int \left(d\vec{X}^\top \wedge *d\vec{X} + \Lambda(\vec{X}^\top \vec{X} - 1) \right).$$

Equations of motion

$$\partial_+ \partial_- \vec{X} + (\partial_+ \vec{X} \cdot \partial_- \vec{X}) \vec{X} = 0.$$

Similarly for AdS^5 embedded into $\mathbb{R}^{2,4}$ with coordinates $\vec{Y}(\tau, \sigma)$.

Virasoro constraints link both sectors

$$(\partial_\pm \vec{X})^2 = (\partial_\pm \vec{Y})^2.$$

Start with (abstract) solution $\vec{X}(\tau, \sigma), \vec{Y}(\tau, \sigma)$.

Lax Pair

Introduce (left=right) $\mathfrak{so}(6)$ current

$$j = -j^\top = 2\vec{X}d\vec{X}^\top - 2d\vec{X}\vec{X}^\top.$$

Flatness, conservation (equations of motion) & Virasoro constraints

$$dj + j \wedge j = 0, \quad d*j = 0,$$

$$\text{Tr } j_\pm^2 = -8(\partial_\pm \vec{X})^2 = -8(\partial_\pm \vec{Y})^2 = \text{Tr } \tilde{j}_\pm^2.$$

Lax pair: Family of flat connections (\leadsto integrability)

$$a(x) = \frac{1}{1-x^2} j + \frac{x}{1-x^2} *j.$$

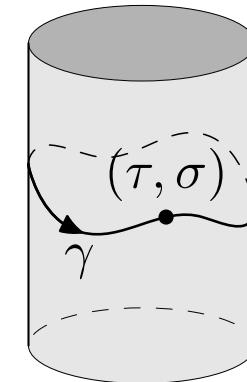
Flatness of $a(x)$ for all x equivalent to flatness and conservation of j :

$$da(x) + a(x) \wedge a(x) = 0.$$

Monodromy

Monodromy of Lax connection around closed string

$$\omega(x) = \text{P exp} \oint_{\gamma} (-a(x)).$$



Independent of path γ , but not of end point $\gamma(0) = \gamma(2\pi) = (\tau, \sigma)$

$$d\omega(x) + [a(x), \omega(x)] = 0.$$

Shift generates similarity transformation. Eigenvalues preserved

$$\omega(x) \simeq \text{diag}(e^{+iq_1(x)}, e^{+iq_2(x)}, e^{+iq_3(x)}, e^{-iq_3(x)}, e^{-iq_2(x)}, e^{-iq_1(x)}).$$

The quasi-momenta $q_k(x)$ are conserved, gauge-invariant quantities.
Complete set of action variables in Hamilton-Jacobi formalism.

Global Charges

Expansion of Lax connection at $x = \infty$:

$$a(x) = -\frac{1}{x} * j + \mathcal{O}(1/x^2), \quad J = \frac{\sqrt{\lambda}}{4\pi} \oint * j.$$

Global $\mathfrak{so}(6)$ charges J can be read off from monodromy at $x = \infty$

$$\omega(x) = I + \frac{1}{x} \frac{4\pi J}{\sqrt{\lambda}} + \mathcal{O}(1/x^2).$$

Expansion of quasi-momentum q for S^5 and \tilde{q} for AdS^5

$$q_k(x) = \frac{1}{x} \frac{4\pi(J_1, J_2, J_3)}{\sqrt{\lambda}} + \dots, \quad \tilde{q}_k(x) = \frac{1}{x} \frac{4\pi(E, S_1, S_2)}{\sqrt{\lambda}} + \dots$$

Fix $q_k(\infty) = \tilde{q}_k(\infty) = 0$.

Local Charges

Poles at $x = \pm 1$: Can diagonalize Lax connection perturbatively

$$u(x, \sigma) (\partial_\sigma + a_\sigma(x, \sigma)) u(x, \sigma)^{-1} = \partial_\sigma - i \sum_{k=-1}^{\infty} (x \mp 1)^k \delta Q_k^\pm(\sigma).$$

Leading charge density $\delta Q_{\pm 1}^\pm(\sigma)$ related to current j_\pm

$$\delta Q_{\pm 1}^\pm \simeq -\frac{i}{2} j_\pm \simeq |\partial_\pm \vec{X}| \operatorname{diag}(+1, 0, 0, 0, 0, -1).$$

Diagonalized current gives conserved local charges Q_k^\pm

$$q_1(x) = \sum_{k=-1}^{\infty} (x \mp 1)^k Q_k^\pm, \quad Q_k^\pm = \oint_0^{2\pi} \delta Q_{k,11}^\pm(\sigma).$$

Residues of $q_1(x)$ and $\tilde{q}_1(x)$ linked by Virasoro constraint $Q_{-1}^\pm = \tilde{Q}_{-1}^\pm$.

Inversion Symmetry

Map $x \mapsto 1/x$ relates Lax connection of right and left currents j, ℓ

$$h(d + a_j(x))h^{-1} = d + a_\ell(1/x), \quad h = 1 - 2\vec{X}\vec{X}^\top.$$

Both are equal $j = \ell$ in this model, therefore $x \mapsto 1/x$ is **symmetry**

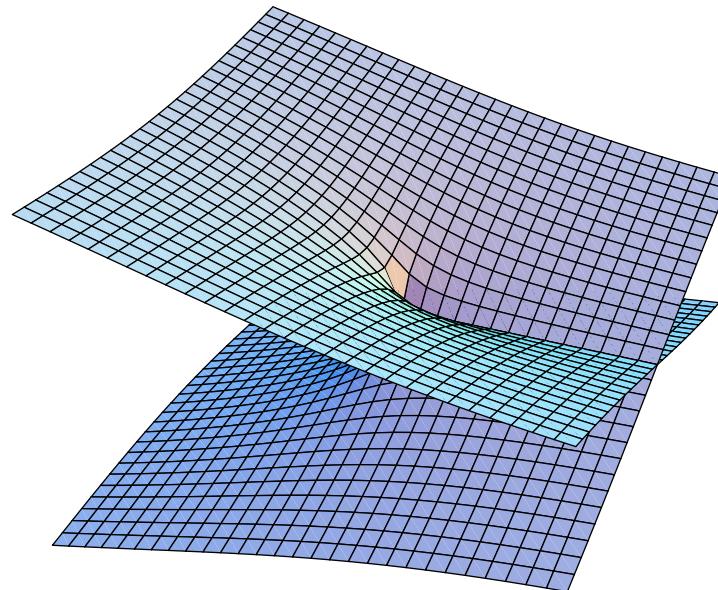
$$\omega(1/x) = h(2\pi)\omega(x)h^{-1}(0) \simeq \omega(x).$$

Consistent choice for quasi-momenta

$$q_1(1/x) = 4\pi n_0 - q_1(x), \quad q_{2,3}(1/x) = q_{2,3}(x).$$

Analyticity

Monodromy $\omega(x)$ is analytic in x except at $x = \pm 1$: $\omega(x) \sim \exp \frac{iQ_{-1}^{\pm}}{x \mp 1}$.
Diagonalization introduces new **singularities** $\{x_a^*\}$ (eigenvalue crossing)

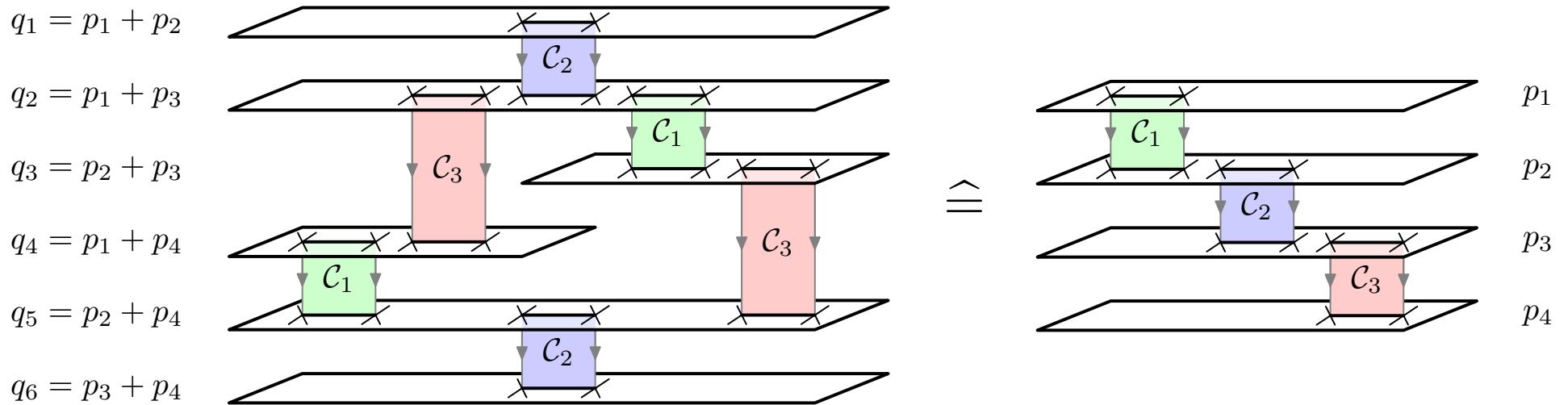


One full turn around x_a^* interchanges eigenvectors/values (labelling).
Generic behaviour at degenerate eigenvalues $e^{iq_k(x_a^*)} = e^{\pm iq_l(x_a^*)}$:

$$e^{iq_k(x)}, e^{\pm iq_l(x)} = e^{iq_k(x_a^*)} \left(1 \pm \alpha_a \sqrt{x - x_a^*} + \mathcal{O}(x - x_a^*) \right).$$

Riemann Surface

6 eigenvalues as one function q on a Riemann surface \mathbb{M}_6 with 6 sheets



Eigenvalues of $\omega_6(x)$

$$q_1 + q_6 = q_2 + q_5 = q_3 + q_4 = 0$$

Eigenvalues of $\omega_4(x)$

$$p_1 + p_2 + p_3 + p_4 = 0$$

Lax connection $a_{\mathbf{R}}$ and monodromy $\omega_{\mathbf{R}}$ exist for any representation \mathbf{R} .

Simplify: Use chiral spinor (4) representation of $\mathfrak{so}(6) = \mathfrak{su}(4)$.

$$\omega_4(x) \simeq \text{diag}\left(e^{ip_1(x)}, e^{ip_2(x)}, e^{ip_3(x)}, e^{ip_4(x)}\right).$$

Algebraic Curve

Can the Riemann surface \mathbb{M} be embedded in \mathbb{C}^2 as an algebraic curve?

- Finite genus:

Assume finitely many branch points $\{x_a^*\}$ (low energy). ✓

Other solutions should be considered as limiting cases.

- Eigenvalues $e^{ip(x)}$ of $\omega(x)$ are analytic almost everywhere. ✓
- Branch points $\{x_a^*\}$ are square-root singularities. ✓
- Monodromy $\omega(x)$ has exponential singularities at $x = \pm 1$. ✗
- Quasi-momentum $p(x)$ is defined modulo 2π . ✗
- $p'(x)$ is unique and has only square-root and pole singularities. ✓

$p'(x)$ is the algebraic curve associated to a classical string $\vec{X}(\tau, \sigma)$

$\vec{X}(\tau, \sigma) \implies F(x, p'(x)) = 0$ with F polynomial (degree 4 in p').

Admissible Curves

Not all algebraic curves can arise from the sigma model. Ansatz:

$$F(x, p') = P_4(x) p'^4 + P_3(x) p'^3 + P_2(x) p'^2 + P_1(x) p' + P_0(x).$$

Constraints:

- Sum of solutions vanishes $p'_1 + p'_2 + p'_3 + p'_4 = 0$.
- Inversion symmetry, $p'(1/x) = x^2 p'(x)$.
- Asymptotics $p'(x) \sim 1/x^2$ at $x = \infty$.
- Double poles in $p'(x)$ at $x = \pm 1$. Virasoro constraint. No residues.
- Physical branch points $p'(x) \sim 1/\sqrt{x - x_a^*}$.
- No unphysical branch points $p'(x) \sim \sqrt{x - x^\times}$.
- $p(x) = \int_{\infty}^x p'(x') dx'$ must be single-valued (modulo 2π).

Count moduli of admissible curves.

Polynomials

Use $y(x) = (x - 1/x)^2 x p'(x)$ instead of $p'(x)$ and

$$F(x, y) = P_4(x) y^4 + P_3(x) y^3 + P_2(x) y^2 + P_1(x) y + P_0(x) = 0.$$

- Asymptotics and symmetry

$$P_k(x) = c_{k,0} x^k + c_{k,1} x^{k+1} + \dots + c_{k,1} x^{4A+7-k} + c_{k,0} x^{4A+8-k}.$$

- Physical branch points & sum of solutions:

$$P_4(x) = x^4 \prod_{a=1}^{2A} (x - x_a^*)(x - 1/x_a^*), \quad P_3(x) = 0.$$

- Unphysical branch cuts removed by demanding:

All solutions to $dF(x, y) = 0$ with $x \neq x_a^$ must lie on the curve.*

- Double poles without residue: $F(\pm 1, y) \sim (y^2 - \alpha_\pm^2)^2$, $F'_x(\pm 1, y) = 0$.

Cycles

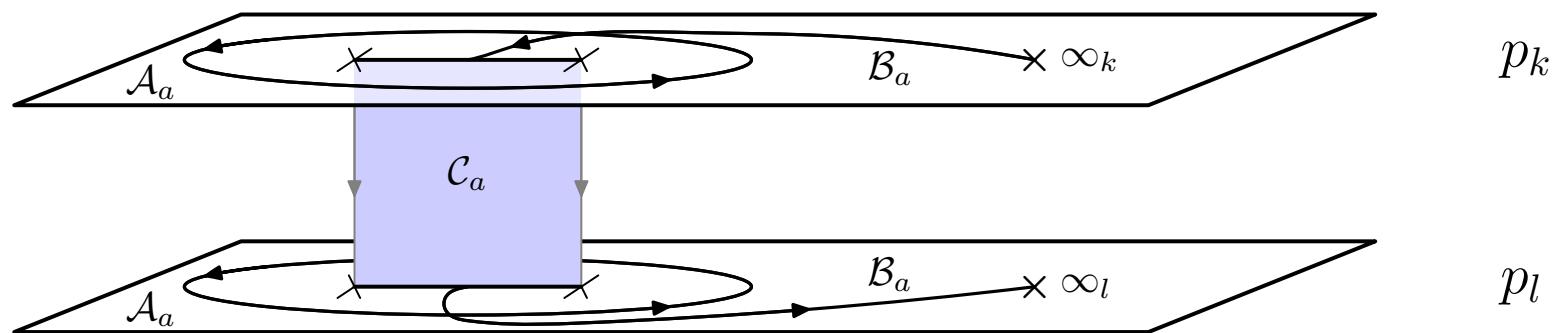
Single valuedness: All closed cycles must be integer

$$\oint dp \in 2\pi\mathbb{Z} \quad \text{as well as} \quad \int_{\infty_k}^{\infty_l} dp \in 2\pi\mathbb{Z} \quad \text{and} \quad \int_{\infty_k}^{0_k} dp \in 2\pi\mathbb{Z}.$$

Can arrange cuts \mathcal{C}_a such that

$$\oint_{\mathcal{A}_a} dp = 0, \quad \int_{\mathcal{B}_a} dp = 2\pi n_a$$

with \mathcal{A}, \mathcal{B} -cycles defined as in



String Moduli

Start with $4A$ branch points, i.e. A cuts + A images under inversion.

- Polynomial $F(x, y) = y^4 x^{4A} + \dots$ has $\approx 20A$ coefficients.
- $\approx 4A$ fixed by $P_3(x) = 0$.
- $\approx 8A$ fixed by inversion symmetry.
- $\approx 5A$ fixed by removing unphysical branch points.
- $\approx 2A$ fixed by integrality of cycles.

Careful counting: Precisely A moduli remain. Conclusions:

One “mode number” $n_a \in \mathbb{Z}$ and one “amplitude” $K_a \in \mathbb{R}$ for each cut

$$n_a = \frac{1}{2\pi} \int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i} \oint_{\mathcal{A}_a} p(x) \left(1 - \frac{1}{x^2}\right) dx.$$

Solutions classified by $\{(k_a, l_a, n_a, K_a)\}$ or if (k_a, l_a, n_a) unique

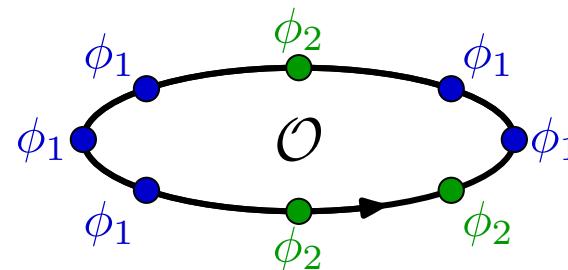
Solutions classified by \vec{K}_n .

Gauge Theory and Spin Chains

Single trace operator, two complex scalars ϕ_1, ϕ_2 (a.k.a. \mathcal{Z}, ϕ or Z, X)

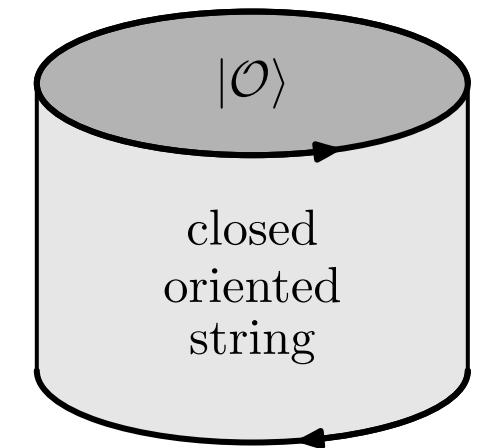
$$\mathcal{O} = \text{Tr } \phi_1 \phi_1 \phi_2 \phi_1 \phi_1 \phi_1 \phi_2 \phi_2$$

Length L : # of fields

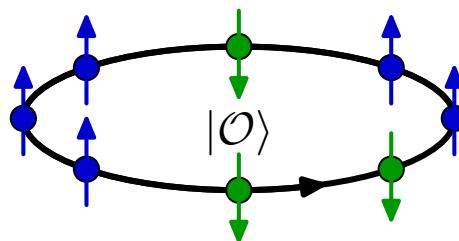


Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$



Length L : # of sites



Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\dots\rangle + *|\dots\rangle + \dots$

Dilatation Generator

Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

$$\mathfrak{D}(g) \mathcal{O} = D_{\mathcal{O}}(g) \mathcal{O}.$$

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2 \mathfrak{D}_2 + g^3 \mathfrak{D}_3 + g^4 \mathfrak{D}_4 + \dots$$

Local action along spin chain (homogeneous)

$$\begin{aligned}
 & \text{Diagram of a local action vertex with four outgoing lines and a central pink circle.} \\
 & = \sum_{p=1}^L \text{Diagram of a spin chain segment from } p-2 \text{ to } p+4 \text{ with a central pink circle at } p. \\
 & \quad \text{The spin chain is shown as a horizontal line with points labeled } p-2, p-1, p, p+1, p+2, p+3, p+4. \\
 & \quad \text{The central pink circle is labeled } \mathcal{O}(x). \\
 & = \sum_{p=1}^L \text{Diagram of a spin chain segment from } p-2 \text{ to } p+4 \text{ with a central pink circle at } p. \\
 & \quad \text{The spin chain is shown as a horizontal line with points labeled } p-2, p-1, p, p+1, p+2, p+3, p+4. \\
 & \quad \text{The central pink circle is labeled } \mathcal{O}(x).
 \end{aligned}$$

One-Loop

One-loop $\mathcal{O}(g^2)$ dilatation operator \mathfrak{D}_2 :

$$\mathfrak{D}_{2(12)} = \text{Diagram with } \mathfrak{D}_2 \text{ loop} = \text{Diagram with red wavy line} + \text{Diagram with red cross} + \frac{1}{2} \text{Diagram with red loop at top} + \frac{1}{2} \text{Diagram with red loop at bottom}$$

Extract logarithmic piece of Feynman diagrams.

- $\mathfrak{so}(6)$ subsector of scalars: $\mathfrak{D}_{2(12)} = I_{(12)} - P_{(12)} - \frac{1}{2}K_{(12)}$.
- Complete dilatation operator of $\mathcal{N} = 4$ SYM [hep-th/0307015]^{NB} [hep-th/0407277]^{NB}

$$\mathfrak{D}_{2(12)} = 2h(J_{(12)}), \quad h(j) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}.$$

$J_{(12)}$: “total superconformal spin” operator.

- Planar dilatation operator integrable.

[Minahan Zarembo]^{NB} [Kristjansen Staudacher]^{NB} [Staudacher]

Bethe Ansatz

Bethe equations for $\mathfrak{so}(6)$ sector. Bethe roots: $a_k, b_k, c_k \in \mathbb{C}$.

[Minahan
Zarembo]

$$\prod_{j=1}^{K_a} \frac{a_k - a_j + 2i}{a_k - a_j - 2i} \prod_{j=1}^{K_b} \frac{a_k - b_j - i}{a_k - b_j + i} = -1,$$

$$\prod_{j=1}^{K_a} \frac{b_k - a_j - i}{b_k - a_j + i} \prod_{j=1}^{K_b} \frac{b_k - b_j + 2i}{b_k - b_j - 2i} \prod_{j=1}^{K_c} \frac{b_k - c_j - i}{b_k - c_j + i} = -\frac{(b_k + i)^L}{(b_k - i)^L},$$

$$\prod_{j=1}^{K_b} \frac{c_k - b_j - i}{c_k - b_j + i} \prod_{j=1}^{K_c} \frac{c_k - c_j + 2i}{c_k - c_j - 2i} = -1.$$

Momentum constraint and one-loop scaling dimension:

$$\prod_{j=1}^{K_b} \frac{b_j + i}{b_j - i} = 1, \quad D_2 = \sum_{j=1}^{K_b} \frac{4}{b_j^2 + 1}.$$

Similar for complete $\mathcal{N} = 4$ SYM: 7 types of Bethe roots.

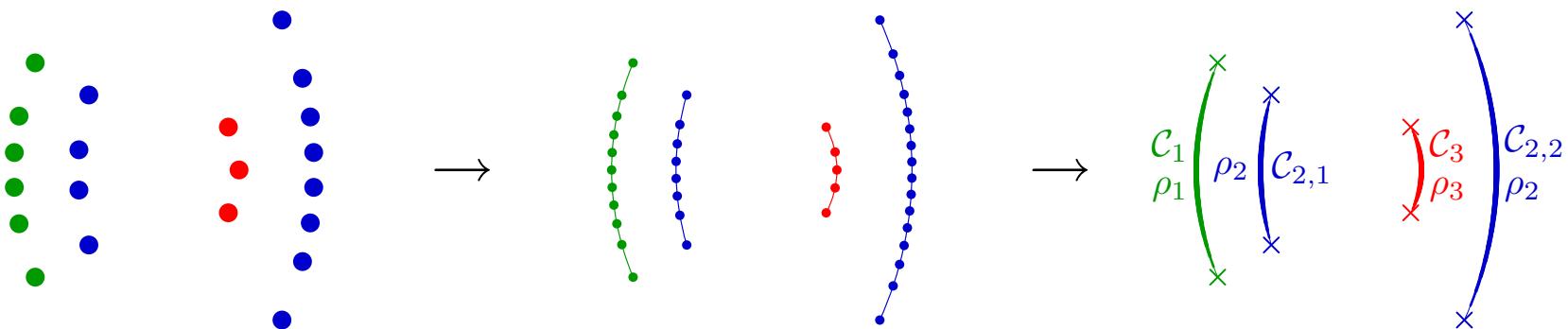
[NB
Staudacher]

Thermodynamic Limit

- Long spin chains, $L \rightarrow \infty$.
- Large number of Bethe roots $K_a, K_b, K_c \sim L$.
- Low energy, $D_2 \sim 1/L$.

[NB, Minahan
Staudacher
Zarembo]

Roots a_k, b_k, c_k condense on (disconnected) contours $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$:



Discrete sums turn into integrals ($\prod = \exp \sum \log$) with densities ρ_1, ρ_2, ρ_3

$$\sum_{k=1}^{K_b} f(b_k) \rightarrow \int_{\mathcal{C}_2} du \rho_2(u) f(u).$$

Bethe Equations in the Thermodynamic Limit

Bethe equations in thermodynamic limit become integral equations

$$\begin{aligned}
 +4 \int_{\mathcal{C}_1} \frac{dv \rho_1(v)}{v-u} - 2 \int_{\mathcal{C}_2} \frac{dv \rho_2(v)}{v-u} &= 2\pi n_{1,a} \quad \text{for } u \in \mathcal{C}_{1,a} \\
 -2 \int_{\mathcal{C}_1} \frac{dv \rho_1(v)}{v-u} + 4 \int_{\mathcal{C}_2} \frac{dv \rho_2(v)}{v-u} - 2 \int_{\mathcal{C}_3} \frac{dv \rho_3(v)}{v-u} + \frac{2}{u} &= 2\pi n_{2,a} \quad \text{for } u \in \mathcal{C}_{2,a} \\
 -2 \int_{\mathcal{C}_2} \frac{dv \rho_2(v)}{v-u} + 4 \int_{\mathcal{C}_3} \frac{dv \rho_3(v)}{v-u} &= 2\pi n_{3,a} \quad \text{for } u \in \mathcal{C}_{3,a}.
 \end{aligned}$$

Momentum constraint and one-loop scaling dimension:

$$2 \int_{\mathcal{C}_2} \frac{du \rho_2(u)}{u} = 2\pi n_0, \quad D_2 = \frac{4}{L} \int_{\mathcal{C}_2} \frac{du \rho_2(u)}{u^2}.$$

Algebraic Curve

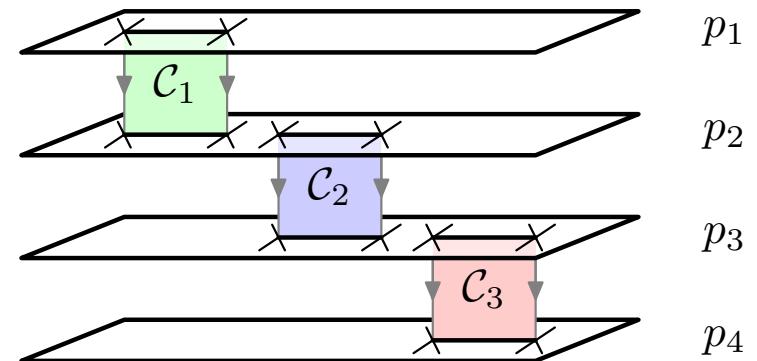
Define a function $p(u)$ with four sheets $p_k(u)$

$$p_1(u) = 2 \int_{\mathcal{C}_1} \frac{dv \rho_1(v)}{v - u} + \frac{1}{u},$$

$$p_2(u) = 2 \int_{\mathcal{C}_2} \frac{dv \rho_2(v)}{v - u} - 2 \int_{\mathcal{C}_1} \frac{dv \rho_1(v)}{v - u} + \frac{1}{u},$$

$$p_3(u) = 2 \int_{\mathcal{C}_3} \frac{dv \rho_3(v)}{v - u} - 2 \int_{\mathcal{C}_2} \frac{dv \rho_2(v)}{v - u} - \frac{1}{u},$$

$$p_4(u) = - 2 \int_{\mathcal{C}_3} \frac{dv \rho_3(v)}{v - u} - \frac{1}{u}.$$



- By construction: $p_k(u)$ analytic and single-valued, $\int_{\mathcal{A}_a} dp = 0$. ✓
- Bethe equations: $p(u)$ discontinuous through cuts, $\int_{\mathcal{B}_a} dp \in 2\pi\mathbb{Z}$. ✓
- At $u = \infty$: Global charges, $p_k(u) \sim (J_1, J_2, J_3)/x + \dots$ ✓
- At $u = 0$: Local charges & No inversion symmetry. SM: $x = \pm 1$. ✗
- One-loop gauge theory as limit of sigma model curve, $u \sim x/L$. ✓✓

Conclusions & Outlook

★ Classical Bosonic Strings on $AdS_5 \times S^5$

- Analytic properties of monodromy of Lax connection.
- Construction of the algebraic curve.
- Classification of solutions.

★ One-Loop $\mathcal{N} = 4$ SYM

- Bethe equations in the thermodynamic limit.
- Construction of the algebraic curve.

★ Vorsätze für 2005

- Do it for the complete theory (in progress).
- Find higher-loop gauge theory (inspiration from string theory).
- Find quantized string theory (inspiration from gauge theory).