# Stability of Landau-Ginzburg Branes

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# Introduction

Goal: Understand stability conditions for D-branes on Calabi-Yau manifolds

Here: B-model the "category of topological D-branes" using matrix factorizations of Landau-Ginzburg superpotentials

Motivation:

- Characterize the spectrum of BPS branes at a distinguished point in moduli space, hopefully independent of what is going on elsewhere

- Describe moduli spaces of D-branes ($e.g.$, the D0-brane!) in non-geometric phases of Calabi-Yau compactifications

# Plan

- Assume *some* familiarity with general aspects of D-branes on Calabi-Yau spaces

- Review *other* aspects, such as $\Pi$-stability and matrix factorizations

- Collect necessary ingredients which I think are essentially sufficient for the definition of a stability condition for matrix factorizations describing D-branes in Landau-Ginzburg models

- Concrete proposal

- Study examples of stable and unstable matrix factorizations

- Interlude: Index theorem for matrix factorizations

# $\Pi$-Stability

The basis of $\Pi$-stability (Douglas, Aspinwall, Bridgeland) is a simple observation about the conformal field theory on the string worldsheet.

A Calabi-Yau background is, from the worldsheet point of view, an $\mathcal{N} = (2,2)$ superconformal field theory, $\mathcal{C}$, with integral central charge $\hat{c}(= 3)$ and integral worldsheet $U(1)$ R-charges for all closed string fields.

A D-brane in this background is, by definition, a conformally invariant boundary condition, $\mathcal{B}$, for $\mathcal{C}$. BPS branes contain, in addition to the boundary conditions on the $\mathcal{N} = 2$ algebra, a boundary condition on the spectral flow operator

$$\mathcal{S}_L = e^{i\pi\varphi}\mathcal{S}_R$$

$\varphi$ is the grade of brane, here taking values in $\mathbb{R} \bmod 2\mathbb{Z}$.

A standard conformal field theory argument shows that $U(1)$ charges of open strings stretching between two (not necessarily mutually) BPS branes satisfy

$$q = \varphi' - \varphi \bmod \mathbb{Z}$$

The unitarity constraint on the string worldsheet is

$$0 \leq q \leq \hat{c}$$

Given $\mathcal{C}$, understanding the space of possible $\mathcal{B}$ is a difficult problem. The first step is to study the constraints arising from $\mathcal{N} = 2$ worldsheet supersymmetry, or equivalently, BRST invariance in the topologically twisted theory. In a second step, one would like to identify stability conditions under which boundary flow leads to a “satisfactory” IR fixed point.

In relating the physical to the topological worldsheet theory, one identifies ghost number with worldsheet R-charge. For open strings, this requires lifting the grade $\varphi$ to a real number, $i.e.$,

$$n = q + \varphi - \varphi' \, .$$

So for every physical brane $\mathcal{B}$, there is an infinite number of topological branes $B[m]$.

Note that the physical worldsheet R-charge $q$ does not depend on the lift. On the other hand, the ghost number $n$ is integer and is independent of the moduli. Therefore, continuous variations of $\varphi$ (the phase of the BPS central charge) with (part of) the moduli lead to a variation of R-charges of open strings.

$\Pi$-stability is the interpretation of this variation of $q$, together with the unitarity constraint $0 \leq q \leq \hat{c}$ as a change of stability condition on the category of topological branes.

With respect to the B-model, the space of stability conditions can be identified with the stringy Kähler moduli space $\mathcal{M}_k$.

Given the spectrum of B-type BPS branes at a distinguished basepoint $p_*$ in $\mathcal{M}_k$, $\Pi$-stability tells how the spectrum varies as one moves around in $\mathcal{M}_k$. A natural choice for $p_*$ is large volume, where the category of topological branes is described as the derived category of coherent sheaves of the underlying algebraic variety. Near $p_{\mathrm{LV}}$, $\Pi$-stability reduces to $\mu$-stability. (Douglas-Fiol-Römelsberger)

For a non-compact Calabi-Yau manifold, another interesting point is the orbifold point, at which all central charges align. There, $\Pi$-stability can be related to $\theta$-stability for quiver representations. (Douglas-Fiol-Römelsberger, Aspinwall)

In this talk, the chosen basepoint is the Landau-Ginzburg orbifold point in the moduli space of a compact Calabi-Yau manifold.

# Landau-Ginzburg models and matrix factorizations

An $\mathcal{N} = 2$ Landau-Ginzburg model is specified in the bulk by the choice of a worldsheet superpotential $W$, which is a (homogeneous) polynomial in chiral field variables $x_1, x_2, \ldots, x_r$.

When considering Landau-Ginzburg model on a space with boundary, the supersymmetry variation of the superpotential term exhibits a peculiar boundary term. (This is generic for interactions from integrals over part of superspace.) This "Warner problem" used to make it difficult to describe useful boundary conditions in $\mathcal{N} = 2$ Landau-Ginzburg models.

Matrix factorizations provide the solution of the Warner problem. (Kontsevich, Kapustin-Li, Brunner-Herbst-Lerche-Scheuner).

Let $W \in \mathcal{R} = \mathbb{C}[x_1, \ldots, x_r]$ be a polynomial.

A matrix factorization of $W$ is a pair of $N \times N$ matrices $f$, $g$ with polynomial entries satisfying

$$fg = gf = W \cdot id$$

A matrix factorization is called reduced if all entries of $f$ and $g$ have no constant term, $i.e.$, $f(0) = g(0) = 0$.

Matrix factorizations $(f, g)$ and $(f', g')$ are equivalent if they are related by a similarity transformation

$$U_1 f = f' U_2 \qquad U_2 g = g' U_1$$

where $U_1, U_2 \in GL(N, \mathcal{R})$ are invertible matrices with polynomial entries.

Mathematically, stability is the problem of understanding matrix factorizations modulo similarity transformations.

The category $\mathrm{MF}(W)$ of topological branes in the Landau-Ginzburg model has as objects matrix factorizations

$$Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \qquad\qquad Q^2 = W$$

and as morphisms cohomology classes of the boundary BRST operator $D$ acting as

$$D\Phi = Q'\Phi - (-1)^{\Phi}\Phi Q = \begin{pmatrix} 0 & f' \\ g' & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$$

on morphisms $\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of free $\mathcal{R}$ modules.

$\mathrm{MF}(W)$ is a $\mathbb{Z}_2$ graded triangulated category with shift functor

$$Q[1] = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}[1] = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}$$

and cones

$$Q = \begin{pmatrix} Q_1 & 0 \\ T & Q_2 \end{pmatrix}$$

fitting into the triangle

$$\begin{tikzcd} & Q \arrow[dr, "S_1"] & \\ Q_2 \arrow[ur, "S_2"] & & Q_1 \arrow[ll, dashed, "T"] \end{tikzcd} \qquad S_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note: $[1]^2 = 1$, but this will be improved shortly.

# Physical Interpretation

The bulk superpotential can be viewed as the closed string tachyon. Similarly, the matrices $f$ and $g$ can be viewed as open string tachyon configuration between a stack of space-filling branes and anti-branes. The equation $Q^2 = W$ is the condition that this tachyon configuration preserve $\mathcal{N} = 2$ worldsheet supersymmetry.

In the bulk, homogeneity of $W$ is generally considered necessary and sufficient for the existence of a useful IR fixed point.

The extent to which this might be true with boundaries is the subject of this talk.

# Graded Matrix Factorizations

$W$ is homogeneous if there exists an assignement of R-charges $q_i \in \mathbb{Q}$ such that

$$W(e^{i\lambda q_i} x_i) = e^{2i\lambda} W(x_i) \qquad \lambda \in \mathbb{R} \bmod \pi H \mathbb{Z}$$

This is a $U(1)$ action with respect to which $W$ is equivariant. It is natural to require that this $U(1)$ action can be extended to matrix factorizations. This amounts to the existence of a representation

$$\rho : \mathbb{R}/\pi H \mathbb{Z} \to GL^{+}(2N, \mathcal{R}) = GL(N, \mathcal{R}) \times GL(N, \mathcal{R})$$

such that

$$\rho(0, x_i) = \rho(\pi H, x_i) = id_{2N \times 2N}$$
$$\rho(\lambda, x_i) Q(e^{i\lambda q_i} x_i) = e^{i\lambda} Q(x_i) \rho(\lambda, x_i)$$

We have a slightly non-standard group law

$$\rho(\lambda, x_i)\rho(\lambda', e^{i\lambda q_i} x_i) = \rho(\lambda + \lambda', x_i)$$

Under gauge transformations, $Q(x_i) \rightarrow U(x_i)Q(x_i)U(x_i)^{-1}$ with $U \in GL^+(2N, \mathcal{R})$, $\rho$ transforms as

$$\rho^U(\lambda, x_i) = U(x_i)\rho(\lambda, x_i)U(e^{i\lambda q_i} x_i)^{-1}$$

**Note:** This is a $U(1)$ representation on the matrix factorization as a free $\mathcal{R}$ module, not as a $\mathbb{C}$ vector space. If $\rho$ can be diagonalized, then $\rho^U$ is $x_i$-idependent. It is not immediately obvious that this is always possible.

It will become clear later why we have to allow also gauge transformations of non-zero degree which make $\rho$ non-diagonal and $x_i$-dependent.

# Gradability is a topological condition

The generator of the $U(1)$ action

$$R(\lambda, x_i) = -i\partial_\lambda \rho(\lambda, x_i)\rho(\lambda, x_i)^{-1}$$

satisfies

$$EQ + [R, Q] = Q\,,$$

where

$$E = \sum_i q_i x_i \frac{\partial}{\partial x_i}$$

is the "Euler vector field". $W$ being quasi-homogeneous means $EW = 2W$, and therefore, if $Q^2 = W$,

$$\{Q, EQ - Q\} = 0\,,$$

where $\{\cdot,\cdot\}$ is the anticommutator. In other words, $EQ - Q$ defines a class in $H^1(Q)$. The existence of $R$ is the statement that this class is trivial.

The vanishing of $EQ - Q$ is mirror to the vanishing of the Maslov class of Lagrangian submanifolds of symplectic manifolds, which is the necessary condition that the Floer cohomology be $\mathbb{Z}$-graded. (In general, if the symplectic manifold is orientable, Floer cohomology is $\mathbb{Z}_2$-graded.)

Here is an example of a non-gradable matrix factorization

The superpotential $W = x^3 + y^7$ is the simplest example that is not a simple singularity. There is a finite number of families of rank one factorizations (Schappert '85). One of them is

$$f = \begin{pmatrix} x^2 - \lambda y^5 & xy \\ xy + \lambda^2 y^4 & -\lambda x + y^2 \end{pmatrix} \qquad g = \begin{pmatrix} x - \frac{y^2}{\lambda} & \frac{xy}{\lambda} \\ \frac{xy}{\lambda} + \lambda y^4 & -\frac{x^2}{\lambda} + y^5 \end{pmatrix}$$

For $\lambda \neq 0$, this is stably equivalent to

$$\tilde{f} = \begin{pmatrix} -xy^5 & \lambda x y^4 + y^6 & -x^2 \\ y^6 & x^2 - \lambda y^5 & xy \\ -x^2 - \lambda y^5 & yx + \lambda^2 y^4 & -\lambda x + y^2 \end{pmatrix} \qquad \tilde{g} = \begin{pmatrix} \lambda & y & -x \\ y & x & 0 \\ -x & \lambda y^4 & y^5 \end{pmatrix}$$

which is non-reduced, but has a limit as $\lambda \to 0$.

$W$ is homogeneous with $q_x = 2/3$, $q_y = 2/7$, but

$$Ef - f + R_+ f - f R_- = \frac{2}{21}\lambda \begin{pmatrix} -y^5 & 0 \\ 2\lambda y^4 & -x \end{pmatrix} = \frac{2}{21}\lambda \partial_\lambda f\,.$$

is proportional to the marginal deformation of the family, hence a non-trivial cohomology class. For $\lambda \neq 0$, the matrix factorization is not quasi-homogeneous.

A geometric interpretation could relate $(f, g)$ via mirror symmetry with a Lagrangian with (generically) non-vanishing Maslov class.

# Further remarks

- If $R$ exists, it might not be unique (more below)

- Given $R$, the morphism spaces inherit a $\mathbb{Q}$-grading in addition to the $\mathbb{Z}_2$ grading

- The $\mathbb{Q}$ grading is compatible with the triangulated structure

- By orbifolding, one can combine the $\mathbb{Z}_2$ and $\mathbb{Q}$ grading to a single $\mathbb{Z}$-grading, together with a grade of the matrix factorization, $i.e.$,

$$q = \varphi' - \varphi + n$$

- Conjecture: $\mathfrak{MF}(W) \cong \mathbf{D}(X)$, where $X$ is the Calabi-Yau related to $W/\Gamma$ via GLSM ($cf.$, Ashok, Dell'Aquila, Diaconescu)

- Landau-Ginzburg monodromy is shift by $2$ in the $\mathbb{Z}$ graded category

# Ambiguities of $R$

The condition

$$EQ - Q = [Q, R]$$

fixes $R$ up to the addition of an even $Q$ cycle.

To fix the ambiguity, interpret $R$ as a character on the gauge group of similarity transformations (more below). Infinitesimally

$$\delta Q = [V, Q] \qquad \text{with } V \in \mathrm{Mat}^{+}(2N \times 2N, \mathcal{R}), \mathrm{Tr}\, V \in \mathbb{C}$$

$$\chi_R(V) = \mathrm{Tr}(RV)$$

The ambiguities of $R$ can be fixed by the requirement that this character vanish for trivially acting gauge transformations

$$\mathrm{Tr}(RV) = 0 \qquad \text{whenever} \qquad [V, Q] = 0$$

# RR charges and Index Theorem

If matrix factorizations represent D-branes in string theory, they must carry Ramond-Ramond charge.

The RR ground states to which B-branes couple are those with $q_L = -q_R$. In a $\mathbb{Z}_H$ Landau-Ginzburg orbifold, these states can be described as follows. In the $l$-th twisted sector, divide the fields in two groups, the *twisted* ones, $\{x_i^t\}$, with $lq_i \notin 2\mathbb{Z}$, and the *untwisted* ones, $\{x_i^u\}_{i=1,\ldots,r_l}$, with $lq_i \in 2\mathbb{Z}$. The RR ground states with $q_L = -q_R$ from the $l$-th twisted sector surviving the $\mathbb{Z}_H$ projection are in correspondence with the neutral RR ground states of the "effective" LG potential $W_l(x_i^u) = W(x_i^u, x_i^t = 0)$, obtained by setting the twisted fields to zero. In turn, those correspond to the elements of the chiral ring of $W_l$ with $q_L = q_R = \hat{c}^u/2$, and survive the $\mathbb{Z}_H$ projection if and only if $r_l$ is even.

Notation: $\mathcal{H}_{\rm RR}^{\rm B}$ spanned by $|l;\alpha\rangle$ in correspondence with $\phi_l^\alpha$ in chiral ring of $W_l$.
$Q_l = Q(x_i^u, x_i^t = 0)$. Generator of $\mathbb{Z}_H$: $\gamma$

The RR-charge of a matrix factorization $Q$ is given by

$$\mathrm{ch}(Q) : \mathcal{H}^{\mathrm{B}}_{\mathrm{RR}} \to \mathbb{C}$$

$$\begin{aligned}\mathrm{ch}(Q)(|l;\alpha\rangle) &= \langle l;\alpha|Q\rangle_{\mathrm{disk}} \\ &= \frac{1}{r_l!}\mathrm{Res}_{W_l}\left(\phi_l^\alpha \ \mathrm{Str}\left[\gamma^l(\partial Q_l)^{\wedge r_l}\right]\right) = \frac{1}{r_l!}\oint \frac{\phi_l^\alpha \ \mathrm{Str}\left[\gamma^l(\partial Q_l)^{\wedge r_l}\right]}{\partial_1 W_l \cdots \partial_{r_l} W_l}\end{aligned}$$

The residue is normalized by the dimension of chiral ring of $W_l$

$$\mathrm{Res}_l(\det \partial_i \partial_j W_l) = \oint \frac{\det \partial_i \partial_j W_l}{\partial_1 W_l \cdots \partial_{r_l} W_l} = \dim \mathcal{J}_l = \mu_l = \prod_{l q_i \in 2\mathbb{Z}} \frac{2 - q_i}{q_i}$$

# Index Theorem

$$\mathrm{Tr}(-1)^F = \sum_{n\in\mathbb{Z}}(-1)^n \dim \mathrm{Hom}^n_{\mathfrak{MF}(W)}(Q,Q') = \langle \mathrm{ch}(Q'), \mathrm{ch}(Q)\rangle$$

$$= \frac{1}{H}\sum_{l=0}^{H-1}\sum_{\alpha,\beta}\mathrm{ch}(Q')(|l;\alpha\rangle)\frac{1}{\prod_{lq_i\notin 2\mathbb{Z}}(1-\omega_i^l)}\eta_l^{\alpha\beta}\mathrm{ch}(Q)(|l;\beta\rangle)^*$$

$\eta_l^{\alpha\beta}$: the inverse of the closed string topological metric in the $l-th$ twisted sector,

$$\eta^l_{\alpha\beta} = \mathrm{Res}_l\left(\phi_l^\alpha\phi_l^\beta\right)$$

**Proof** available in the case that $r_l = 0$ in all twisted sectors (this includes the quintic).

# Stability

The physical question: Which matrix factorizations flow under boundary worldsheet RG flow, to a single, unitary, boundary conformal field theory?

In situations with spacetime supersymmetry, the BCFT will describe a BPS brane.

Mathematically, stability issues arise in the context of moduli problems. Typical case: Symplectic quotient construction with respect to a reductive algebraic group. The stable orbits admit a distinguished representative at the zeroes of the moment map. This is the BPS object modulo the real gauge group.

(More generally, worldsheet RG flow should split the unstable objects at singular points along the flow into the direct sum of its stable constituents.)

- For a general homogeneous Landau-Ginzburg model, the unitarity constraint

$$0 \leq q \leq \hat{c}$$

and the assertion that worldsheet RG flow leads to a single BCFT in the IR makes sense independent of orbifolding and integrality of central charge.

- For models with a geometric interpretation, this will be a particular stability condition on $\mathbf{D}(X)$. It is distinguished by the fact that it originates in the unorbifolded model. Smaller K-theory $\rightsquigarrow$ more rigid stability condition.

- How to impose unitarity constraint in practice? Physical prescription: (Douglas) Integrate out cancelling brane-antibrane pairs—In other words, restrict to reduced matrix factorizations.

**Note:** This does not lead in an obvious way to an abelian category, which is something one would expect on general grounds (Bridgeland) at least after orbifolding.

# Proposal

Let $W(x_1, \ldots, x_r)$ be a quasi-homogeneous Landau-Ginzburg polynomial, $EW = 2W$, where $E = \sum q_i x_i \partial_i$. Let $Q$ be a reduced quasi-homogeneous matrix factorization of $W$. $Q$ is called R-semistable if in all triangles

$$\begin{tikzcd}
& Q \arrow[dr, "S_1"] & \\
Q_2 \arrow[ur, "S_2"] & & Q_1 \arrow[ll, dashed, "T"]
\end{tikzcd}$$

in which $Q$ participates opposite to the fermionic morphism $T$, we have

$$q_T \leq 1 \quad \Leftrightarrow \quad q_{S_1} \geq 0 \quad \Leftrightarrow \quad q_{S_2} \geq 0 \,.$$

$Q$ is stable if the only triangles for which $q_T = 1$ are those with $Q_1$ or $Q_2$ equal to $Q$ (and the other equivalent to $0$).

Here $q_T$ is defined by the condition

$$ET + R_2T - TR_1 = q_TT$$

## Simple check

With present definitions

$$\mathrm{Hom}^0(Q,Q') \quad = \quad \mathfrak{H}^{q=\varphi'-\varphi}(Q,Q')$$

morphism space
in $\mathbb{Z}$-grading

morphism space
in $\mathbb{Q}$-grading

(when grade of branes is defined) Then,

$$\varphi > \varphi' \Rightarrow \mathrm{Hom}^0(Q,Q') = 0$$

This is one of the conditions in the definition of Bridgeland

# Relation with moduli spaces

Mathematically, the problem is to understand solutions of $Q^2 = W$ modulo the action of the gauge group $G \cong GL^+(2N, \mathcal{R}) \cong GL(N, \mathcal{R}) \times GL(N, \mathcal{R})$. $\rightsquigarrow$ An algebraic group acting on a linear space with a constraint. This is analogous to moduli spaces for quiver representations studied by King.

Let $Y$ be the space of representations of a quiver. The gauge group $\mathcal{G}$ is the product of GL's at the nodes. Stability is defined with respect to a character $\chi : \mathcal{G} \to \mathbb{C}^\times$ via

## Mumford's Numerical Criterion

A representation $y \in Y$ is $\chi$-semistable iff $\chi$ is trivial on the stabilizer of $y$ and if every one-parameter subgroup $g(\lambda) = e^{\lambda a}$ of $\mathcal{G}$, for which $\lim_{\lambda \to \infty} g(\lambda) y$ exists, satisfies $\langle d\chi, a \rangle \geq 0$, where $d\chi$ is the infinitesimal version of $\chi$ evaluated on the generator $a$ of $g(\lambda)$.

Here: All triangles in $\mathrm{MF}(W)$ are isomorphic to the standard cone

$$Q = Q_1 \oplus Q_2 + T\,, \qquad R = R_1 \oplus R_2 + (q_t - 1)\left[\frac{N_2}{N_1+N_2}S_1 - \frac{N_1}{N_1+N_2}S_2\right]$$

where $S_i = id_i$. Under the one-parameter group of gauge transformations generated by $V = S_1$, this cone transforms as

$$Q_\lambda = e^{\lambda V} Q e^{-\lambda V} = Q_1 \oplus Q_2 + e^{-\lambda} T \underset{\lambda\to\infty}{\longrightarrow} Q_1 \oplus Q_2\,.$$

The limit $\lambda \to \infty$ splits the cone back into its constituents. The condition $q_T \leq 1$ is equivalent to

$$-\,\mathrm{Tr}(RV) = -(q_t - 1)\frac{2N_2N_1}{N_2+N_1} \geq 0$$

thus identifying $\mathrm{Tr}(R\,\cdot\,)$ as the character of $G$ with respect to which we are defining stability.

# Remarks on the gauge group

- As a complex Lie group, $G$ is infinite-dimensional
- Degree $0$ gauge transformations are generated by

$$\mathfrak{g}^0 = \{V \in \mathfrak{g}; EV + [R, V] = 0\}$$

$\mathfrak{g}^0$ is non-reductive! Since both polynomial and total degree are preserved in matrix multiplication, $\mathfrak{g}^0$ has a maximal solvable subalgebra consisting of those matrices without constant term.

# Moment map-like flow

For a practical toy model: Introduce a hermitian metric on the space of matrix factorizations

$$Q=\sum_\alpha Q_\alpha x^\alpha\,,\qquad Q'=\sum_\alpha Q'_\alpha x^\alpha\,,\qquad \langle Q,Q'\rangle=\sum_\alpha \mathrm{Tr}\big(Q_\alpha^\dagger Q'_\alpha\big)$$

Restrict to a finite-dimensional subgroup of $G$ and choose a basis of generators $\{V_i\}$. Study the flow

$$\frac{dQ}{dt}=-\big(\langle Q,[V^i,Q]\rangle-\mathrm{Tr}\,RV^i\big)[V_i,Q]$$

**Note:** This is a moment map for the maximal reductive subgroup of the degree $0$ gauge group. As we will see, this gauge group is too small.

## Examples

### Standard cone

$$Q_\lambda = Q_1 \oplus Q_2 + e^{-\lambda} T$$

$$\frac{d\lambda}{dt} = \left(e^{-2\lambda} ||T||^2 + (q_T - 1)\beta\right)$$

with $\beta = 2N_1 N_2/(N_2 + N_1)$.

$\rightsquigarrow$ Stable for $q_T < 1$, unstable for $q_T \geq 1$.

# Brane-Antibrane Annihilation

Let $(f, g)$ be a matrix factorization of $W$ with R-matrix $R = (R_+, R_-)$, and consider the cone over the identity $id : (f, g) \to (f, g)$,

$$Q_0 = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 1 & g \\ g & 0 & 0 & 0 \\ -1 & f & 0 & 0 \end{pmatrix} .$$

This is gauge equivalent to direct sums of the trivial factorization $W = 1 \cdot W$ via the gauge transformation

$$U_\lambda = \begin{pmatrix} 1 & -\lambda f & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda g \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad U_1 Q_0 U_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & -W \\ 0 & 0 & 1 & 0 \\ 0 & W & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

What is the correct R-matrix? Starting from the cone, we would guess

$$R^{\mathrm{cone}} = \begin{pmatrix} R_+ - \frac{1}{2} & 0 & 0 & 0 \\ 0 & R_- + \frac{1}{2} & 0 & 0 \\ 0 & 0 & R_- - \frac{1}{2} & 0 \\ 0 & 0 & 0 & R_+ + \frac{1}{2} \end{pmatrix}$$

This does not satisfy $\mathrm{Tr}(RV) = 0$ for trivially acting gauge transformations, and does not transform into $\mathrm{diag}(-1/2, 1/2, -1/2, 1/2)$ under $U_1$.

The "correct" choice is

$$R_0 = \begin{pmatrix} -\frac{1}{2} & R_+ f - f R_- & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & g R_+ - R_- g \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(The correctness of this choice is more obvious if we consider not only brane-antibrane.)

But this does not assign degree $0$ to the gauge transformation $U_\lambda$!

Conclusion: Gauge transformations of non-zero degree are naturally important to describe brane-antibrane annihilation.

The naive flow (which makes sense only with $R_0$!) leads to $\lambda = 1$, as expected.

# Stable and unstable flows

Consider superpotential $W = x^5 + y^5$, and factorization

$$f_{\text{unst}} = \begin{pmatrix} x & y & 0 \\ 0 & x^3 & y \\ y^3 & 0 & x \end{pmatrix} \qquad g_{\text{unst}} = \begin{pmatrix} x^4 & -xy & y^2 \\ y^4 & x^2 & -xy \\ -x^3y^3 & y^4 & x^4 \end{pmatrix}$$

$$R = \text{diag}(7/10, -1/10, -1/10, 1/10, 1/10, -7/10)$$

This factorization is unstable. There is a morphism of negative degree $q = -\frac{1}{10}$ with the tensor product of minimal models ($\hat{c} = 6/5$). Study the decay into minimal model branes via

$$F_{\text{unst}} = \begin{pmatrix} x & y & 0 & 0 \\ -y^4 & x^4 & 0 & 0 \\ 0 & -x^3 & x^4 & -y \\ y^3 & 0 & y^4 & x \end{pmatrix}, \qquad \text{with corresponding } G_{\text{unst}}$$

Compare the flow of $(F_{\mathrm{unst}}, G_{\mathrm{unst}})$ with that of $(F_{\mathrm{stab}}, G_{\mathrm{stab}})$

$$F_{\mathrm{stab}} = \begin{pmatrix} x & y^2 & 0 & 0 \\ -y^3 & x^4 & 0 & 0 \\ 0 & -x^3 & x^4 & -y^2 \\ y & 0 & y^3 & x \end{pmatrix}, \qquad G_{\mathrm{stab}}$$

which to best knowledge is stable and equivalent to

$$f_{\mathrm{stab}} = \begin{pmatrix} x & y^2 & 0 \\ 0 & x^3 & y^2 \\ y & 0 & x \end{pmatrix}, \qquad g_{\mathrm{stab}} = \mathrm{adj}(f_{\mathrm{stab}})$$

$\rightsquigarrow$ Study the flow on associated 12-parameter gauge orbits.

# Results

- Starting from generic initial conditions, the flow splits $(F_{\mathrm{unst}}, G_{\mathrm{unst}})$ into minimal model branes.

- On the other hand, $(F_{\mathrm{stab}}, G_{\mathrm{stab}})$ flows to the reduced configuration $(f_{\mathrm{stab}}, g_{\mathrm{stab}})$ and a copy of the trivial brane $(1, W)$.

- Caveat: Because the gauge group is not reductive, the flow is not convex. There are other stationary points on the gauge orbits. But these are unstable in the dynamical sense.

- Other example: $W = x^h + y^3$,

$$f = \begin{pmatrix} x^n & y & 0 \\ 0 & x^n & y \\ y & 0 & x^{h-2n} \end{pmatrix}$$

These are unstable when $n < h/6$ and stable otherwise.

# Summary

- Matrix factorizations are a particularly tractable model for D-branes in Calabi-Yau backgrounds

- Clarified the gradability and grading of matrix factorizations. Relevance of gauge transformations of non-zero degree

- The Landau-Ginzburg model appears to admit a notion of R-stability that is independent of what is going on in the rest of the moduli space

- A naive moment map-like flow works well in several examples