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Recent developments in AdS/CFT and Sasaki-Einstein Geometry

hep-th/0403002, "Sasaki-Einstein Metrics on $S^2 \times S^3$", with J. Gauntlett, D. Martelli, D. Waldram.

hep-th/0411238, "Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals", with D. Martelli.

hep-th/0411264, "An Infinite Family of Superconformal Quiver Gauge Theories with Sasaki-Einstein Duals", with S. Benvenuti, S. Franco, A. Hanany, D. Martelli.

hep-th/0503183, "The Geometric Dual of a-maximisation for Toric Sasaki-Einstein Manifolds", with D. Martelli, S. -T. Yau.

Result: An infinite number of non-trivial AdS/CFT duals $Y^{p,q}$ where both gauge theory and gravity dual are known explicitly $(p, q \in \mathbb{N},\ q < p)$.

Combines recent progress in 3 areas:

- Sasaki-Einstein geometry
- D-branes at toric Calabi-Yau singularities
- a-maximisation in $\mathcal{N} = 1$ superconformal field theories

Setting: Type IIB string theory on the Freund-Ruben background $AdS_5 \times Y_5$. This is supersymmetric if Y_5 is a Sasaki-Einstein manifold.

Definition Y is *Sasaki-Einstein* if its metric cone $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s^2(Y)$ is Ricci-flat and Kähler *i.e.* a Calabi-Yau cone.

Y is then a positively curved Einstein manifold admitting a solution to the Killing spinor equation.

The AdS/CFT correspondence conjectures this background to be dual to a superconformal field theory in four dimensions.

e.g.

$Y_5 = S^5 \leftrightarrow \mathcal{N} = 4\ SU(N)$ Yang–Mills theory

$Y_5 = T^{1,1} \leftrightarrow \mathcal{N} = 1\ SU(N) \times SU(N)$ gauge theory

Sasaki-Einstein geometry

Canonical unit norm Killing vector field $K = J(\partial/\partial r)$ at $r = 1 \rightarrow$ *foliation* of Y. K is called the Reeb vector field. Dual to the R-symmetry in the field theory.

Dual 1-form $\theta \rightarrow$ locally

$$\mathrm{d}s^2(Y) = \theta \otimes \theta + \mathrm{d}s^2(V)$$

where $\mathrm{d}s^2(V)$ is a transverse positive Kähler-Einstein metric.

Orbits of K:

close $\rightarrow U(1)$ action on Y

• free $\rightarrow$ circle bundle over Kähler-Einstein manifold.

Classified in dimension 5: S^5, $T^{1,1}$, $k\#S^2 \times S^3$, $3 \leq k \leq 8$.

• locally free $\rightarrow$ circle orbibundle over Kähler-Einstein orbifold.

e.g. finite quotients of above, Boyer and Galicki examples using existence arguments.

don't close $\rightarrow$ generic orbits densely fill a torus $\mathbb{T}^r$ - think of an irrational line in $\mathbb{T}^2$.

No examples until $Y^{p,q}$. In fact, Cheeger and Tian conjectured in a paper in 1994 that irregular Sasaki-Einstein manifolds do not exist.

Such Sasaki-Einstein manifolds are referred to as regular, quasi-regular, and irregular.

$Y^{p,q}$ metrics

$$\begin{aligned} \mathrm{d}s^2(Y^{p,q}) &= \frac{1-y}{6}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)+\frac{1}{w(y)q(y)}\mathrm{d}y^2 \\ &+ \frac{q(y)}{9}(\mathrm{d}\psi-\cos\theta\mathrm{d}\phi)^2+w(y)\ell^2\,(\mathrm{d}\gamma+A)^2 \end{aligned}$$

- $w(y)$, $q(y)$ certain rational functions of y.
- $\ell = \frac{q}{3q^2-2p^2+p(4p^2-3q^2)^{1/2}}$
- Explicit inhomogeneous metrics that "interpolate" between $\mathbb{Z}_p$ orbifolds of $S^5/\mathbb{Z}_2$ $(q=p)$ and $T^{1,1}$ $(q=0)$.
- $\mathrm{Isom}(Y^{p,q}) \sim SU(2)\times U(1)^2$
- $Y^{p,q} \cong S^2\times S^3$ for $\mathrm{hcf}(p,q)=1$
- $K = 3\frac{\partial}{\partial\psi} - \frac{1}{2\ell}\frac{\partial}{\partial\gamma}$
- Quasi-regular iff $\ell\in\mathbb{Q}$ iff $4p^2-3q^2=n^2$, otherwise irregular, rank 2.
- $\mathrm{vol}(Y^{p,q}) = \frac{q^2[2p+(4p^2-3q^2)^{1/2}]}{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}\pi^3$
- $\mathrm{vol}(T^{1,1}/\mathbb{Z}_p) > \mathrm{vol}(Y^{p,q}) > \mathrm{vol}(S^5/\mathbb{Z}_2\times\mathbb{Z}_p)$

$Y^{p,q}$ are toric Sasaki-Einstein manifolds. This is crucial for constructing the dual superconformal field theory explicitly.

Take (X, ω) a symplectic manifold. Let V be a *symplectic* vector field $\to \mathcal{L}_V \omega = 0 \to \mathrm{d}(V \lrcorner \omega) = 0$.

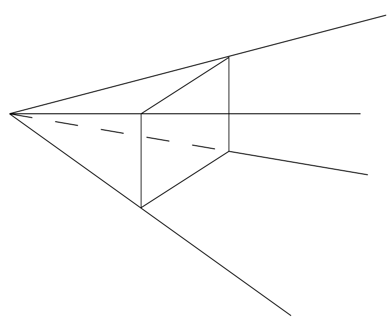
Provided $[V \lrcorner \omega] = 0 \in H^1(X) \to V \lrcorner \omega = \mathrm{d}\mu$, where μ is called the Hamiltonian function for V.

Given a Hamiltonian torus action $\mathbb{T}^n$ on X with vector fields V^i we may use (μ^i, ϕ^i) as coordinates on X, ϕ^i =coordinate along the orbit of V^i. X is then said to be a symplectic toric manifold.

We are interested in the case that $X = C(Y)$ is a toric Calabi–Yau cone. For $Y^{p,q}$ one finds

$$\vec{\mu} = r^2 \left[\frac{1-y}{6}(\cos\theta - 1), \frac{1-y}{6}\cos\theta - \frac{p+q}{2}\ell y, \ell y \right].$$

• The image $\vec{\mu}(X)$ is a strictly convex rational polyhedral cone $\mathcal{C} \subset \mathbb{R}^3$.

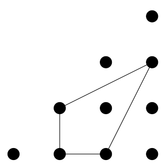
Example:  4-faceted $\mathcal{C}$ in $\mathbb{R}^3$

• X is a compactification of $\mathcal{C}_{\text{interior}} \times \mathbb{T}^3$.

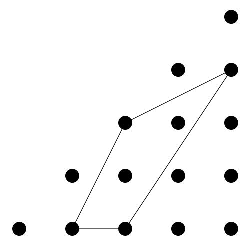
• 1-cycles of $\mathbb{T}^3$ collapse over the facets of $\mathcal{C}$ to give $\mathbb{T}^2$. The *normal vector* to the facet determines the collapsing 1-cycle.

For $Y^{p,q}$, $\mathcal{C}$ has 4 facets, hence 4 normal vectors $v_1 = [1, 0, 0]$, $v_2 = [1, 1, 0]$, $v_3 = [1, p, p]$, $v_4 = [1, p-q-1, p-q]$.

Note that v_a lie in the plane $\mu^1 = 1$. This is a consequence of the Calabi-Yau condition $c_1(X) = 0 \rightarrow$ we may represent a toric Calabi-Yau cone as a convex lattice polytope in $\mathbb{Z}^2$:

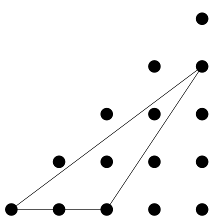


$Y^{2,1}$



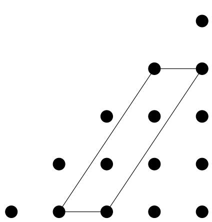
$Y^{3,1}$

- Special limits: $p = q$ and $q = 0$



$Y^{3,3}$

$\mathbb{C}^3/\mathbb{Z}_{2p}$ with action $(z_1, z_2, z_3) \rightarrow (\omega_{2p} z_1, \omega_{2p} z_2, \omega_{2p}^{-2} z_3)$



$Y^{3,0}$

conifold/$\mathbb{Z}_p$ with action $(z_1, z_2, z_3, z_4) \rightarrow (\omega_p z_1, \omega_p^{-1} z_2, \omega_p^{-1} z_3, \omega_p z_4)$ where the conifold is the quadric $z_1 z_2 = z_3 z_4$ in $\mathbb{C}^4$.

Each facet $\mathcal{F}_a$ of the cone $\mathcal{C}$ (vertex of the lattice polytope) is the image of a conic complex toric submanifold D_a of X. Thus $D_a = C(\Sigma_a)$, where explicitly we have

$$\Sigma_1 \cong S^3/\mathbb{Z}_{p-q}, \qquad \Sigma_3 \cong S^3/\mathbb{Z}_{p+q}, \qquad \Sigma_2 \cong \Sigma_4 \simeq S^3/\mathbb{Z}_p$$

N.B. $C(Y^{2,1})$ is the complex cone over the first del Pezzo surface (*not* Kähler-Einstein).

Quiver gauge theories from Calabi-Yau singularities

• N D3-branes transverse to a Calabi-Yau singularity $\rightarrow$ $\mathcal{N}=1$ gauge theory on their worldvolume.

• In the far infra-red, should flow to a superconformal field theory.

• For toric Calabi-Yaus, this gauge theory may be taken to be a toric quiver gauge theory (idea: $C(Y)$ may then be obtained by partial toric crepant resolution of an abelian orbifold of $\mathbb{C}^3$).

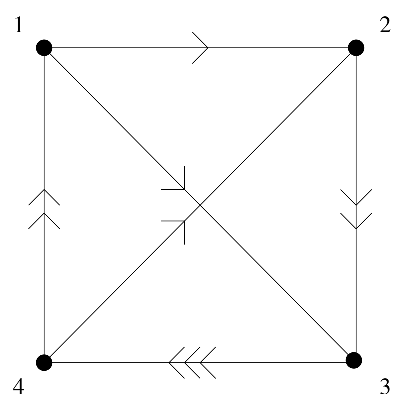
Quiver gauge theories

• directed graph with χ nodes, node $\bullet = U(N)$ gauge group.

• directed edge from node i to node j represents a bifundamental field in the $\bar{N}$ representation of gauge group i, N representation of gauge group j.

Gauge anomaly cancellation $\rightarrow$ # incoming edges = # outgoing edges at each node.

e.g. $C(Y^{2,1})$ is the complex cone over the first del Pezzo surface.



Interactions in this theory are specified by a polynomial superpotential W. May construct gauge invariant terms from closed paths in the quiver: $\mathrm{Tr}(X_1 \ldots X_m)$ from the path $X_1 \rightarrow \ldots \rightarrow X_m$.

e.g. $W = \mathrm{Tr}\left[\epsilon_{\alpha\beta} X_{34}^{\alpha} X_{41}^{\beta} X_{13} + \epsilon_{\alpha\beta} X_{23}^{\alpha} X_{34}^{\beta} X_{42} - \epsilon_{\alpha\beta} X_{12} X_{23}^{\alpha} X_{34}^{3} X_{41}^{\beta}\right]$

The vacuum moduli space (Higgs branch) of such a gauge theory is determined by setting the D terms and F terms to zero, and dividing out by gauge transformations. This moduli space is essentially the Calabi-Yau that the D-branes are probing.

Set $N = 1$ (single D-brane):

• Regard X_{ij} as coordinates on $\mathbb{C}^D$, with torus action by $\mathbb{T}^\chi$ specified by the quiver diagram. Then setting the D terms to zero, mod gauge transformations, is the symplectic quotient $\mathbb{C}^D//\mathbb{T}^\chi$.

• For *toric* quiver theories, every field appears precisely twice in W, with opposite sign, which ensures that setting the F terms to zero d$W = 0$ gives a set of monomial relations among the X_{ij}.

The result is a toric variety, which is the dual Calabi-Yau.

Alternatively: suppose one has an exceptional collection of sheaves generating the derived category for the fully resolved Calabi-Yau variety $\tilde{X}$. This collection has dimension χ = Euler number of $\tilde{X}$. Then each node in the quiver is an exceptional sheaf (fractional brane), and the directed edges are associated to Ext groups between sheaves (which physically count massless modes of strings stretching between the fractional branes).

Gauge theory for $Y^{p,q}$

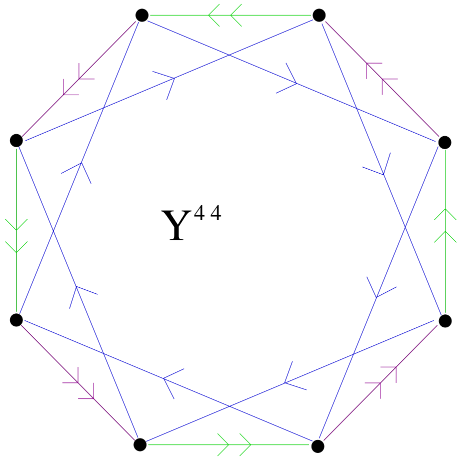
• $X = C(Y^{p,q})$ can be obtained by partial resolution of $\mathbb{C}^3/\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1}$, although this is not computationally practical.

• One easily shows that $\chi(\tilde{X}) = 2p$ which is twice the area of the lattice polytope. Hence the gauge theory has gauge group $U(N)^{2p}$.

An iterative procedure starting from $Y^{p,p}$

Recall $Y^{p,p} = S^5/\mathbb{Z}_{2p}$

e.g. $Y^{4,4}$



This quiver is obtained from standard orbifold techniques. Orbifolds of $\mathbb{C}^3$, in the context of D-brane probes, first studied in 1996 by Sardo Infirri, then Douglas, Greene, Morrison.

$SU(2) \times U(1)$ isometry $\to$ flavour symmetry of the gauge theory.

$2p$ Y_i singlets, p U_i^α doublets, p V_i^α doublets

$$W = \text{Tr}\left[\sum_{i=1}^{p} \epsilon_{\alpha\beta}(U_i^\alpha V_i^\beta Y_{2i+2} + V_i^\alpha U_{i+1}^\beta Y_{2i+3})\right]$$

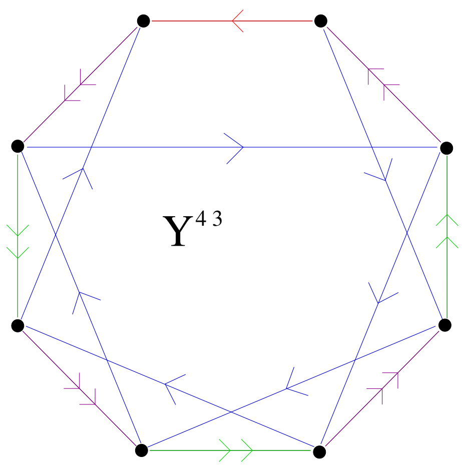
Now pick an (arbitrary) arrow corresponding to a V doublet

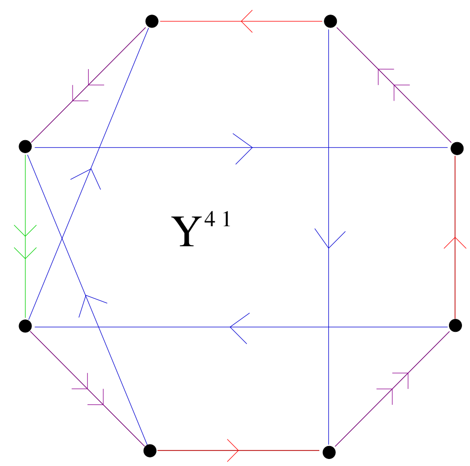
1) replace a V doublet with a new singlet Z

2) remove the two diagonal singlets Y

3) add a new singlet Y to form a loop of length 4.

e.g. This results in the quiver





- # fields $6p \rightarrow 6p - 2$
- In W: -2 cubic terms $\rightarrow$ +1 quartic $\rightarrow$ $12p - 4$ fields

- Repeating the procedure $p - q$ times gives a $Y^{p,q}$ quiver:

fields $= 6p - 2(p-q) = 4p + 2q = p$ U doublets $+$ q V doublets $+$ $(p-q)$ Z singlets $+$ $(p+q)$ Y singlets

- $W = \sum^{2q} UVY + \sum^{p-q} ZUYU$

- Note that this prescription isn't always unique *e.g.* $Y^{4,2}$ leads to two quivers. These are always related by Seiberg duality [Benvenuti, Hanany, Kazakopoulos] and hence are physically equivalent theories in the infra-red. Mathematically, they are related by a mutation of the exceptional collection.

- Note that the gauge theory for $Y^{p,0}$ obtained in this way is indeed the $\mathbb{Z}_p$ quotient of the conifold theory.

a-maximisation

• Any 4 dimensional $\mathcal{N}=1$ SCFT must have a "$U(1)_R$" R-symmetry, which is part of the superconformal algebra $SU(2,2|1)$.

• Constraints: by definition W has R-charge 2, and of course the R-symmetry must be anomaly free.

• The global symmetry group of a SCFT may contain additional flavour symmetries. The abelian part can in principle mix with $U(1)_R$, and thus the above constraints are not sufficient to determine the R-symmetry.

Consider the potential R-symmetry

$R_{\text{trial}} = R_0 + \sum_I s^I F_I$

where F_I are generators of the flavour $U(1)$ symmetries and $s^I \in \mathbb{R}$. R_0 is a fiducial R-symmetry.

• Knowledge of the exact R-charges determines the central charges

$$a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R) \qquad\qquad c = \frac{1}{32}(9\text{Tr}R^3 - 5\text{Tr}R)$$

$$< T^\mu_\mu >= c(\text{Weyl})^2 + a(\text{Euler})^2$$

• According to [Intriligator, Wecht], the exact R-charges are those that locally maximise the central charge a_{trial} as a function of s^I.

The R-charges in $4d$ $\mathcal{N}=1$ SCFTs are thus generally algebraic numbers.

One can impose the above in practise as follows:

- Vanishing of the β-functions at each node is equivalent to anomaly cancellation.

- Each term in the superpotential W has R-charge 2

This is a system of linear equations for the trial R-charges. For the $Y^{p,q}$ quiver gauge theories the space of solutions is always 2-dimensional:

$R[Z]=x$, $R[Y]=y$, $R[V]=1+\frac{1}{2}(x-y)$, $R[U]=1-\frac{1}{2}(x+y)$

$$\frac{32}{9N^2}a(x,y)=2p+(p-q)(x-1)^3+(p+q)(y-1)^3-\frac{p}{4}(x+y)^3+\frac{q}{4}(x-y)^3$$

- Locally maximising $a(x,y)$ gives the R-charges and a central charge of the $Y^{p,q}$ quiver theory at its infra-red fixed point.

$$a_{max}=\frac{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}{4q^2[2p+(4p^2-3q^2)^{1/2}]}N^2$$

Field	$R-$charge	$U(1)_B$
Y	$(-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2})/3q^2$	$p-q$
Z	$(-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2})/3q^2$	$p+q$
U	$(2p(2p-\sqrt{4p^2-3q^2}))/3q^2$	$-p$
V	$(3q-2p+\sqrt{4p^2-3q^2})/3q$	q

Comparison to geometry: According to the AdS/CFT correspondence, the volume of Y_5 is related to the central charge $a(Y_5)$ of the dual gauge theory via

$$\frac{\pi^3N^2}{4\text{vol}(Y_5)}=a(Y_5)$$

Dibaryons and supersymmetric submanifolds

From a bifundamental field A one can construct the dibaryon:

$$\mathcal{B}[A] = \varepsilon^{\alpha_1 \dots \alpha_N} A_{\alpha_1}^{\beta_1} \dots A_{\alpha_N}^{\beta_N} \varepsilon_{\beta_1 \dots \beta_N}$$

$\mathcal{B}[Z]$, $\mathcal{B}[Y]$ are *singlets* of $SU(2)$

$\mathcal{B}[V]$, $\mathcal{B}[U]$ transform in the $\mathbf{N+1}$ of $SU(2)$

- The R-charges are simply $N \cdot R[A]$.

In the geometry dibaryons are D3-branes wrapped on supersymmetric 3-submanifolds Σ. Here supersymmetric means that the metric cone $C(\Sigma)$ is a divisor in the Calabi-Yau cone. The R-charges are given by the AdS/CFT formula:

$$R[\mathcal{B}_a] = \frac{\pi N}{3\,\mathrm{vol}(Y^{p,q})} \mathrm{vol}(\Sigma_a)$$

Recall that we identified 3 supersymmetric cycles in $Y^{p,q}$:

$$\begin{aligned}
\Sigma_1 \simeq S^3/\mathbb{Z}_{p-q} \quad &\rightarrow \quad \mathcal{B}[Z] \\
\Sigma_2 \simeq \Sigma_4 \simeq S^3/\mathbb{Z}_p \quad &\rightarrow \quad \mathcal{B}[U] \\
\Sigma_3 \simeq S^3/\mathbb{Z}_{p+q} \quad &\rightarrow \quad \mathcal{B}[Y] \\
\Sigma_5 \equiv -\Sigma_1 - \Sigma_3 \quad &\rightarrow \quad \mathcal{B}[V]
\end{aligned}$$

- One can verify that the volumes agree with the field theory computation of R-charges.

- One can also check that baryonic and flavour $U(1)$ charges agree.

The dual of a-maximisation: Z-minimisation

From a geometric viewpoint, the a-maximisation calculation just explained is mysterious. How are these volumes being computed?

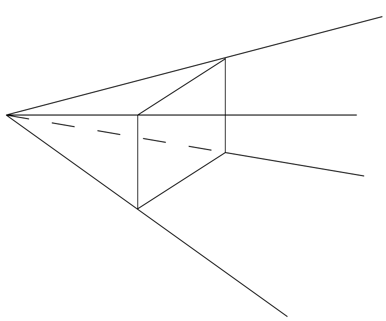
Recall that for a regular Sasaki-Einstein manifold Y, we have $U(1) \hookrightarrow Y \to V$ where V is a positive Kähler-Einstein manifold. Then (for the canonical bundle)

$$\mathrm{vol}(Y) = \frac{2\pi^n}{n^n(n-1)!} c_1^{n-1}[V]$$

A topological invariant. Clearly though the volume for irregular manifolds is not topological.

In a-maximisation one extremises a relatively simple function over a space of trial R-symmetries. In AdS/CFT the R-symmetry is dual to the Reeb vector field K. Motivated by physics, we seek a similar geometrical problem for K.

Set-up: We begin with a toric Gorenstein singularity. For our purposes this is a toric Kähler cone $X = C(Y) \cong \mathbb{R}^+ \times Y$ with $c_1(X) = 0$. Recall that the image under the moment map μ is a convex rational polyhedral cone in $\mathbb{R}^n$:



This is specified by the inward primitive normal vectors v_a, $a = 1, \ldots, d$.
$c_1(X) = 0 \to$ may take $v_a = (1, w_a)$

Denote by (y_i, ϕ_i) the symplectic coordinates on $X \to y_i$ coordinates on $\mathbb{R}^n$.

• Any toric Kähler metric on X may be written

$$\mathrm{d}s^2 = G_{ij}\mathrm{d}y_i\mathrm{d}y_j + G^{ij}\mathrm{d}\phi_i\mathrm{d}\phi_j$$

where $G_{ij} = \partial^2 G(y)/\partial y_i \partial y_j$ for some strictly convex function $G(y)$, which is the Legendre transform of the Kähler potential [Guillemin, 1994].

The function $G(y)$ has a certain prescribed singular structure at $\partial\mathcal{C}$ in order for the metric on X to be smooth.

• X is a cone iff $G_{ij}(y)$ is homogeneous degree -1. The Reeb vector $K = J(r\partial/\partial r)$ is

$$K = b_i \frac{\partial}{\partial \phi_i} \qquad b_i = 2G_{ij}y_j$$

• X is Ricci-flat iff G satisfies the Monge-Ampère equation

$$\det(G_{ij}) = \exp(-2\partial G/\partial y_1)$$

Moreover, imposing that the metric on X is smooth gives $b_1 = n$.

• Y, the base of X, is a $\mathbb{T}^n$-fibration over the characteristic hyperplane

$$2b_i y_i = 1$$

which with $\mathcal{C}$ forms a convex polytope $\Delta = \Delta_b$.

• We note that

$$\mathrm{vol}(Y) = 2n(2\pi)^n \mathrm{vol}(\Delta)$$

$$\mathrm{vol}(\Sigma_a) = (2n-2)(2\pi)^{n-1}\frac{1}{|v_a|}\mathrm{vol}(\mathcal{F}_a)$$

These depend only on the Reeb vector b.

The Einstein-Hilbert action for (Y, h) is

$$S[h] = \int_Y \left[R_Y(h) + 2(n-1)(3-2n)\right] \sqrt{h}\ \mathrm{d}^{2n-1}x$$

which has Euler-Lagrange equation

$$\mathrm{Ric}_Y(h) = (2n-2)h$$

After some calculation in the above formalism, for Sasakian metrics h one computes

$$Z[b] \equiv \frac{1}{(n-1)(2\pi)^n} S[h] = (b_1 - (n-1))2n\ \mathrm{vol}(\Delta)$$

That is, the Einstein-Hilbert action depends only on the Reeb vector b. Thus Sasaki-Einstein metrics are critical points

$$\partial Z / \partial b_i = 0$$

In fact one can prove that necessarily $b_1 = n$ (required for regularity of the Monge-Ampère equation). And then Z is essentially the volume of the Sasakian metric.

• One can prove generally that there always exists a unique critical point of Z, which is a local minimum.

e.g. Take complex dimension $n = 3$. One computes

$$Z = \frac{(b_1 - (n-1))2n}{48 b_1} \sum_a \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}$$

where $v_1, v_2, \ldots, v_d, v_{d+1} \equiv v_1$ are the ordered normals to the polyhedral cone $\mathcal{C}$ and $(\cdot, \cdot, \cdot)$ denotes a 3×3 determinant.

• For the $Y^{p,q}$ toric singularities we precisely recover the earlier results obtained both from the explicit metrics and the gauge theory.

• We are also able to compute other toric Sasaki-Einstein manifolds and find agreement with gauge theory results, where known (*e.g.* the complex cone over the second del Pezzo surface).