Motivations

"Separation of Variables" for classical systems (SoV)

(M, ω) symplectic manifold

$$(p_1, \dots, p_n, q_1, \dots, q_n) \qquad \omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_i$$

Hamilton Jacobi $\quad H\left(q_1, \dots, q_n, \frac{\partial S}{\partial p_1}, \dots, \frac{\partial S}{\partial p_n}\right) = E$

It is called separable in $(\vec{p}, \vec{q})$ if HJ admits complete integral

$$S(\vec{q}, \alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} S_i(q_i, \alpha_1, \dots, \alpha_n)$$

Complete means a solution

$$S = S(\vec{q}; \alpha_1, \dots, \alpha_n)$$

α_i parameters $\quad \left| \frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} \right| \neq 0$

Integrable system $(H_1, \dots, H_n)$

$$H_i : M \longrightarrow \mathbb{R}$$

$$\{H_i, H_j\} = 0$$

$$dH_1 \wedge \dots \wedge dH_n \neq 0$$

is separable in coordinates $(\vec{p}, \vec{q})$

if we have non-trivial relations

$$\Phi_i(\underbrace{q_i, p_i}_{\text{come in pairs}}; H_1, \dots, H_n) = 0 \qquad i = 1, \dots, n$$

$$p_i = p_i(q_i; H_1, \dots, H_n) \qquad i = 1, \dots, n$$

$$S = S(\vec{q}; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} \int^{q_i} p_i(q_i'; H_1, \dots, H_n)\Big|_{H_i = \alpha_i} dq_i'$$

NOTE:

① $H_i = \alpha_i$ define Lagrangian foliation $\mathcal{F}$

② $dS = \theta|_{\mathcal{F}}$

] Geometric (Intrinsic)

Not geometric [

③ Φ_i's have the special property to contain a single pair of canonical coordinates at time.

PROBLEM Find intrinsic conditions on $(H_1, \dots, H_n)$ to ensure separability in a set of coordinates.

Bihamiltonian Approach

Definition A C^∞ manifold M is bihamiltonian

$P_1, P_2 \in \Gamma(\Lambda^2 TM)$ Poisson tensors

$(P_i(df, dg) = \{f, g\}_i \quad i = 1, 2)$

① $P_1 + \lambda P_2$ is Poisson $\forall \lambda \in \mathbb{R}\ (\mathbb{C})$

② P_2 is invertible

$$N = P_1 P_2^{-1} \in \Gamma(\mathrm{End}\,TM)$$

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) \equiv 0$$

$$\forall X, Y \in \Gamma(TM)$$

Theorem On M we have coordinates (x_i, y_i)

① $P_2 = \omega = \sum_{i=1}^{n} dy_i \wedge dx_i$ ② $N = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dx_i \right)$

λ_i are functions on M (eigenvalues of N)

MOREOVER the only non zero Poisson brackets are

$$\{x_i, y_j\}_2 = \delta_{ij}$$

$$\{x_i, y_j\}_1 = \lambda_i \delta_{ij}$$

Theorem The system $(H_1, .., H_n)$ is separable in $(x_i, y_i) \Longleftrightarrow \{H_i, H_j\}_1 = 0 = \{H_i, H_j\}_2$

$$\forall\, i,j = 1, .., n.$$

Different Approach

Hamiltonian Systems admitting a Lax represent. with spectral parameter (+ r-matrix formulation)

Separation coordinates are "provided" by the spectral curve

$$\det(\mu I - L(\lambda)) = 0$$

$L(\lambda)$ = Lax Matrix

It is possible to find (sometime)

$(\lambda_1, .., \lambda_n, \mu_1, .., \mu_n)$ on the phase space such that:

1. $P_i = (\lambda_i, \mu_i)$ belongs to the spectral curve, $i = 1 \dots n$
2. the hamiltonian system is separable in these coordinates.

Abel Jacobi map

$$
\begin{tikzcd}
\mathcal{S} \arrow[rr, leftrightarrow, "I"] \arrow[dr] & & \mathbb{J} \arrow[dl, "\pi"] \\
& \mathcal{B} &
\end{tikzcd}
$$

Relative Jacobian

Family of curves

Theorem (Donagi–Freed, Markman–Witten)

1. There is holomorphic symplectic form on $\mathbb{J}$ such that π is lag.

$\Longleftrightarrow \exists (z_1, \dots, z_n)$ coordinates on $\mathcal{B}$

local, holomorphic

$$\tau_{ij} = \frac{\partial^2 F}{\partial z_i \partial z_j}$$

holomorphic function on $\mathcal{B}$

period matrix of $\mathbb{J}$

2. Equivalently: $\mathcal{B}$ is special-Kähler

$\mathcal{B}$ is Kähler ω, I, g

$\nabla : \Gamma(T\mathcal{B}) \to \Gamma(T^*\mathcal{B} \otimes T\mathcal{B})$

$\nabla \omega = 0 \quad T_\nabla = 0 \quad R_\nabla = 0$

$d_\nabla I = 0$

- Compare the two methods (settings)

- How much we can recover of both of Them from a Lagrangian fibration? (Classical Arnold-Liouville I. S.)

Theorem (Bartocci - M)

If $\pi : X \to B$ is a Lagrangian fibration, B symplectic such that $\nabla \Omega = 0$ (Ω symplectic form on B, ∇ Linear connec. on TB) then X is (Locally) Hypercomplex / Hypersymplectic, B is (Locally) Special Kähler

$(X, \overset{\omega}{\Omega})$ symplectic manifold $\dim_{\mathbb{R}} X = 2N$

$\pi : X \to B$ fibration $\quad \pi^{-1}(b) = F_b$ cpt connect.

$\Omega|_{F_b} \equiv 0$

(F_b are tori)

For each $b \in B$

$C_b = \{$ invariant vector fields along $F_b \}$
(F_b carries an $\mathbb{R}^n$- action)

$C_b \supset \Gamma_b = \{$ invariant vector fields of period equal to 1 $\}$

$$\Gamma_b \cong H_1(F_b, \mathbb{Z})$$
$$\lambda \longmapsto [\gamma(\lambda)]$$

lattice in C_b

Integral curve of λ

Define a sheaf on B, $\Gamma = \{$period lattice of $\pi : X \to B\}$

$$\Gamma \otimes_{\mathbb{Z}} C^\infty_B \cong C = \left\{ \begin{array}{c}\text{Invariant vertical} \\ \text{Vector fields}\end{array} \right\}$$

Vector bundle on B

$$\pi^* C \cong \mathrm{Vert}(TX)$$

$U \subset B$ open set $b \in U$

$\{X_i\}$ basis for $\Gamma|_U \rightsquigarrow \{\gamma_i(b)\}$ basis for $H_1(F_b, \mathbb{Z})$

$$dI_i = X_i \lrcorner \omega$$

X_i vertical invariant $\rightsquigarrow$ I_i hamiltonian functions constant along fibers $\rightsquigarrow$ Functions on U $(I_1, \dots, I_n)$

Can Find $\{\varphi_i\}$ functions on F_b (smoothly variable w.r.to $b \in B$) $\frac{1}{2\pi} \int_{\gamma_i} d\varphi_i = \delta^i{}_k$

Induces: $(I_1, \dots, I_N, \varphi_1, \dots, \varphi_N)$

Action-Angle coordinates on $U \subset B$

$$T^*B \simeq C$$

$$dI_i \rightsquigarrow X_i$$

$$TB \cong R^1\pi_*\mathbb{R} \otimes_{\mathbb{R}} C^\infty_B$$

$$\nabla : \Gamma(TB) \longrightarrow \Gamma(TB \otimes T^*B)$$

$$\nabla := 1 \otimes d$$

<u>Proposition</u>: ∇ is flat and torsion free

$$\nabla^2 = 0$$

$\left\{\frac{\partial}{\partial I_i}\right\}_{i=1,\dots,n}$ provides with a $\mathbb{Z}$-basis for $\Gamma^* = \mathrm{Hom}(\Gamma, \mathbb{Z}) \cong R^1\pi_*\mathbb{Z}$

This gives us an $\mathbb{R}$-basis for $R^1\pi_*\mathbb{R}$:

$\Rightarrow$ $\frac{\partial}{\partial I_j} \mapsto X_j \otimes 1$ (This is the $\simeq$ $TB \simeq R^1\pi_*\mathbb{R} \otimes C^\infty_B$

$$\nabla\left(\frac{\partial}{\partial I_j}\right) = (1 \otimes d)(X_j \otimes 1) = 0$$

$\Rightarrow$ Connection is torsion free

NOTE:

$$\begin{array}{ccc} C & \cong & T^*B \\ \cup & & \cup \\ \Gamma & \rightsquigarrow & \Lambda \end{array}$$

(Lagrangian covering of B loc, freely generated by closed 1-forms)

Monodromy of Λ coincide with holonomy of ∇

Suppose $X \xrightarrow{\pi} B$ has a section

Proposition: ∇ induces the following splitting:

$$TX \cong \pi^*(T^*B) \oplus \pi^*(TB)$$

① $\pi : X \to B$ has a section $\Rightarrow$ $T^*B/\Lambda \simeq TX$ (discrete quot.)

$\Downarrow$

a) $T^*B \xrightarrow{\rho} X$ (proj)

b) $\begin{tikzcd} T^*B \arrow[r, "s"] \arrow[rr, bend right, "p"'] & X \arrow[r, "\pi"] & B \end{tikzcd}$

c) $\rho^*(TX) \cong TT^*B$

② ∇ on T^*B $\Rightarrow$ $0 \to Vert(TT^*B) \to TT^*B \to p^*TB \to 0$

$$TT^*B \simeq Vert(TT^*B) \oplus p^*(TB)$$

③ $\pi : X \to B$ is lagrangian $\Rightarrow$ $Vert(TX) \simeq \pi^*(T^*B)$

$$\rho^*(TX) \simeq \rho^*\mathrm{Vert}(TX) \oplus (\rho^* \circ \pi^*)TB \simeq (\rho^* \circ \pi^*)T^*B \oplus (\rho^* \circ \pi^*)TB$$

Consequence: ∇ inducer a connection on TX

T is a real Torus $\qquad \hat{T} = H^1(T;\mathbb{R}) / H^1(T;\mathbb{Z})$

Relative analog.

$\pi : X \to B$ $\qquad \hat{\pi} : \hat{X} \to B$

Torus fibration

$$\hat{X} = R^1\pi_*\mathbb{R} / R^1\pi_*\mathbb{Z} \cong TB/\Lambda^*$$

(If we start with lag. fibration

<u>NOTE</u>: $\hat{X} \xrightarrow{\hat{\pi}} B$ has always a lag. section

<u>Theorem</u>: $\hat{X}$ ~~$\hat{M}$~~ is complex manifold

$T\hat{X}$ ~~$T\hat{M}$~~ $\cong \pi^*(TB) \oplus \pi^*(TB)$

$$J : T\hat{X} \to T\hat{X}$$
$$(\alpha, \beta) \mapsto (-\beta, \alpha)$$

is almost complex. Integrability follows from compatibility with ∇ (flat and torsion free).

Local coordinates: $(I_1, .., I_n, \Phi_1, .., \Phi_n)$

$\{\Phi_i\}$ are dual coordinates to

$\{\varphi_i\}$

$z_k = I_k + i\,\Phi_k$ are holomorphic

w.r. to J.

NOTE: If $\begin{array}{c} X \\ \downarrow \pi \\ B \end{array}$ has a section

① $X \simeq \hat{X}$ (as real Torus fib)

② the connection on X induced by the one on TB will coincide with the one coming from

$$T\hat{X} \simeq \pi^* TB \oplus \pi^* TB$$

③ $\{\Phi_i\} \leadsto \{\varphi_i\}$ under the isomorphism $X \simeq \hat{X}$

We will work locally

1) Suppose B is symplectic, Ω symplectic form

2) We suppose $\sigma: B \to X$ (section) or

$$\sigma_\omega : B \longrightarrow X$$

Lagrangian section

(Locally is always true because Arnold-Liouville Th.)

3) $\nabla \Omega = 0$

$$T^*B \overset{\Omega}{\simeq} TB$$

$$0 \to \mathrm{Vert}(TM) \to TM \to \pi^*(TB) \to 0$$

$$TM \overset{\nabla}{\simeq} \pi^*(TB) \oplus \mathrm{Vert}(TM) \overset{(\Omega,\omega)}{\cong} \pi^*(T^*B) \oplus \pi^*(T^*B)$$

$\Rightarrow \quad \underset{\text{vertical}}{\mathcal{X} = (-\Omega^\vee)} \oplus \underset{\text{hor.}}{\Omega}$ symplectic form on X

Since $\nabla\Omega = 0 \Rightarrow \underline{x} = (x_1, .., x_n, y_1, .., y_n)$

are flat symplectic

$\Omega = dx \wedge dy \qquad \mathcal{X} = -dp \wedge dq + dx \wedge dy$

$\underline{\varphi} = (p_1, .., p_n, q_1, .., q_n)$

(are vertical coordinates in X)

Lemma: The symplectic form

χ and ω (on X)

can be written as:

① $\omega = dq \wedge dJ - dp \wedge dx + dq \wedge dy$

$(p, q) = \{\varphi_i\}$ $\quad p_i = \varphi_i \quad i = 1, \dots, n$

$q_i = \varphi_{i+n} \quad i = 1, \dots, n$

② $\chi = -dp \wedge dq + dx \wedge dy$

On $\hat{X}$:

$J_\omega = dx \otimes \frac{\partial}{\partial P} - dP \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial y}$

$J_\chi = -dP \otimes \frac{\partial}{\partial Q} + dQ \otimes \frac{\partial}{\partial P} + dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$

(which is defined only locally)

$(\underbrace{x, y}_{\{I_i\}}, \underbrace{P, Q}_{\{\Phi_i\}})$

Define : $K = J_\omega \circ J_x$

Proposition ① K is complex structure on $T\hat{X}$

② holomorphic coordinates

$$(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$$

$$d\alpha = dx + i\, dQ \qquad d\beta = dP + i\, dy$$

On X we have

$$\sigma = dp \wedge dx + dy \wedge dy$$

$$K = -dx \otimes \frac{\partial}{\partial Q} + dQ \otimes \frac{\partial}{\partial x} - dP \otimes \frac{\partial}{\partial y} + dy \otimes \frac{\partial}{\partial P}$$

$$=: J_\sigma$$

Theorem: $X \xrightarrow{\pi} B$ (real) Lag. fib

(B,Ω) $\nabla\Omega = 0$

then the choice of a section of π over $U \subseteq B$ defines hypersymplectic structure on $X|_U$ and hypercomplex structure on $\hat{X}|_U$

$$X|_U \simeq \hat{X}|_U$$

$$(x,y,\underbrace{p,q}_{\varphi}) \rightsquigarrow (x,y,\underbrace{P,Q}_{\Phi})$$

$$J_\omega = dx \otimes \frac{\partial}{\partial p} - dp \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial y}$$

Induces $(z, w) = (x + ip, y + iq)$ local hol. coordinates (on X)

Fix $\quad s_\sigma : B \to X$ lagrangian section (locally it is pos.)

w.r to σ

$s_\sigma(B) \subseteq X$ is complex submanifold w.r to J_ω

MOREOVER: $J_\omega \big|_{s_\sigma(B)}$ induces

$$I \in \mathrm{End}(TB)$$

$$I^2 = -Id$$

Local coordinates: $I = -\left(dp \otimes \frac{\partial}{\partial x} + dq \otimes \frac{\partial}{\partial y}\right)$

Lemma: $d_\nabla I = 0$

(x, y) are flat coordinates w.r.to ∇.

Theorem (B, Ω, I, ∇) is pseudo special Kähler.