Stringy Corrections to Spacetime Superpotentials in Heterotic Strings

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In recent years there's been renewed interest in string compactifications as approaches to real-world phenomenology (KKLT, landscapes...). Much of the interest has revolved around type II compactifications with fluxes and (anti-)branes.

At the same time, there's been a smaller community interested in understanding analogous issues in heterotic string compactifications. To compactify a heterotic string, one needs to specify not only an underlying space (possibly with B field fluxes), but also a nonabelian gauge field over that space, satisfying certain conditions.

Historically that nonabelian gauge field has made heterotic strings complicated to understand.

For one example, for a susy heterotic string compactification, the nonabelian gauge field must define a holomorphic vector bundle, and furthermore must satisfy the “Donaldson-Uhlenbeck-Yau” PDE

$$g^{i\bar{\jmath}} F_{i\bar{\jmath}} \; = \; 0$$

(close to large radius), descending ultimately from demanding that the susy variation of the 10D gaugino vanish.

Metric dependence of the DUY PDE:

- On a Kähler manifold, the Kähler cone splits into a finite number of subcones, with different moduli space in each subcone
- Over a cpx nonKähler mfld, either
  - **no** gauge fields satisfying the DUY PDE,
  - the Gauduchon cone (replacing the Kähler cone) breaks into infinitely many subcones, such that even infinitesimal metric variations will break the DUY PDE.

(D term constraints)

Nonperturbative corrections to spacetime superpotentials:

(This will be the subject of today's talk.)
Roughly, two sources:

- Gauge instantons – sometimes a 't Hooft effective action will correct a tree-level superpotential. (See *e.g.* Dine-Seiberg's old work on $N = 1$ susy in 4d.)
- Worldsheet instantons – these arise from strings wrapping minimal-area 2-cycles in the spacetime, known as "holomorphic curves."

I'll concentrate on the latter class.

In type II strings, worldsheet instanton corrections have a long history and are well-understood nowadays. (Gromov-Witten, *etc*)

However, the heterotic analogues of these computations, for the most part, have not been developed.

Classes of superpotential terms

To be specific, imagine that we have compactified a heterotic string on a CY 3-fold with a rank 3 vector bundle, breaking an E_8 to E_6, and so the low-energy theory contains $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s in addition to singlets.

- *Charged matter couplings e.g.* $\overline{\mathbf{27}}^3$*.* When the gauge bundle = tangent bundle, these are computed by the "A model topological field theory," and correspond to Gromov-Witten invariants, essentially. For more general gauge bundles, no tech exists.

(cont'd ....)

Classes of superpotential terms

- *Charged matter couplings e.g.* $\mathbf{27}^3$*.* When the gauge bundle = tangent bundle, these are computed by the "B model topological field theory," and are purely classical – no nonperturbative (in α') corrections. For more general gauge bundles, no tech exists.
- *Gauge singlet matter couplings.* When the gauge bundle = tangent bundle, the previous two classes could be computed by well-known math tricks, but no such tricks exist for gauge singlets. Turns out that in "many" cases, individual worldsheet instanton contributions are nonzero but cancel out when you add them all up, resulting in no net nonperturbative correction. (Dine-Seiberg-Wen-Witten, Silverstein-Witten, Candelas *et al*, Beasley-Witten)

What I'm going to talk about today are nonperturbative corrections to *e.g.* $\overline{\mathbf{27}}^3$ and, later, $\mathbf{27}^3$ couplings.

(Why only nonperturbative? Why no perturbative corrections in α'? Answer: forbidden by Kähler axion.)

There's another, more formal, motivation for what I'll describe today, namely: (0,2) mirror symmetry.

Ordinary mirror symmetry: $X_1 \leftrightarrow X_2$, X_1, X_2 CY's

(0,2) mirror symmetry: $(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$ where $\mathcal{E}_1$, $\mathcal{E}_2$ are bundles on X_1, X_2

(0,2) mirror symmetry is poorly understood at present.

Recently, Adams-Basu-Sethi studied (0,2) mirrors. They applied old work of Morrison-Plesser, more recently explained by Hori-Vafa, to (0,2) GLSM's, to make some predictions for (0,2) mirrors in some relatively simple cases.

They also made some predictions for analogues of $\overline{\mathbf{27}}^3$ superpotential terms, or equivalently product structures in heterotic chiral rings, which we verified, and is part of the subject of today's talk.

Outline

- review A model TFT, half-twisted (0,2) TFT
- review correlation f'n computations in A model, describe analogue for (0,2) models
  - formal structure similar; (0,2) generalizes A model
  - compactification issues; not only $\mathcal{M}$, but bundles on $\mathcal{M}$
- apply GLSM's; not only naturally compactify $\mathcal{M}$, but also naturally extend the bundles
- Adams-Basu-Sethi prediction
- Analogue for B model
- Consistency conditions in closed string B model

As outlined before, when the gauge bundle = tangent bundle, the $\overline{\mathbf{27}}^3$ and analogous couplings are computed by a 2d TFT called the "A model."

What I'll be describing amounts to a (0,2) analogue or generalization of the ordinary A model.

First: what is the A model?

The 2D TFT's are obtained by changing the worldsheet fermions: worldsheet spinors $\mapsto$ worldsheet scalars & vectors.

Concretely, that means if we start with the nonlinear sigma model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} \,+\, ig_{i\bar{\jmath}}\psi_-^{\bar{\jmath}}D_z\psi_-^i \,+\, ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\overline{z}}\psi_+^i \,+\, R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}$$

then we deform the $D\psi$'s by changing the spin connection term. Since $J \sim \overline{\psi}\psi$, this amounts to making the modification

$$L \,\mapsto\, L \,\pm\, \frac{1}{2}\omega J \quad\Longleftrightarrow\quad T \,\mapsto\, T \,\pm\, \frac{1}{2}\partial J$$

More formally (useful for computation), A model:

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} \;+\; i g_{i\bar{\jmath}}\psi_-^{\bar{\jmath}} D_z \psi_-^i \;+\; i g_{i\bar{\jmath}}\psi_+^{\bar{\jmath}} D_{\overline{z}} \psi_+^i \;+\; R_{i\bar{\jmath}k\bar{l}}\psi_+^i \psi_+^{\bar{\jmath}} \psi_-^k \psi_-^{\bar{l}}$$

$$\psi_-^i (\equiv \chi^i) \;\in\; \Gamma((\phi^* T^{0,1} X)^\vee) \qquad \psi_+^i (\equiv \psi_z^i) \;\in\; \Gamma(K \otimes \phi^* T^{1,0} X)$$

$$\psi_-^{\bar{\imath}} (\equiv \psi_{\overline{z}}^{\bar{\imath}}) \;\in\; \Gamma(K \otimes \phi^* T^{0,1} X) \qquad \psi_+^{\bar{\imath}} (\equiv \chi^{\bar{\imath}}) \;\in\; \Gamma((\phi^* T^{1,0} X)^\vee)$$

Massless states

Since we no longer have worldsheet *spinors*, we no longer sum over spin sectors. In effect, the only surviving sector in this field theory is the RR sector of the original theory.

So, part of the 2D TFT story is that we're only considering RR sectors (consistent b/c no worldsheet spinors).

So, massless spectrum computations are done in RR sector only. Otherwise, proceed much as usual – states are built as Q-invariant objects, where Q is a subset of susy, which in fact corresponds to the scalar fields.

Massless states

Under the scalar supercharge,

$$\begin{aligned} \delta\phi^i &\propto \chi^i, & \delta\phi^{\bar{\imath}} &\propto \chi^{\bar{\imath}} \\ \delta\chi^i &= 0, & \delta\chi^{\bar{\imath}} &= 0 \\ \delta\psi^i_z &\neq 0, & \delta\psi^{\bar{\imath}}_{\bar{z}} &= 0 \end{aligned}$$

States (Q-cohomology):

$$\begin{aligned} \mathcal{O} \sim b_{i_1\cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \chi^{i_1} \cdots \chi^{i_p} &\leftrightarrow H^{p,q}(X) \\ Q &\leftrightarrow d \end{aligned}$$

The A model TFT is, first and foremost, still a QFT.

But, if you only consider correlation functions between Q-invariant massless states, then the correlation functions reduce to purely zero-mode computations – (usually) no meaningful contribution from Feynman propagators or loops, and the correlators are independent of insertion positions.

Some of this you've seen elsewhere – eg in 4d $\mathcal{N} = 1$ susy models, correlation functions involving, roughly, products of chiral operators, are independent of insertion position (Cachazo-Douglas-Seiberg-Witten). (Basic pt: worldsheet deriv $\propto \overline{Q}^{\dot{\alpha}}$ commutators, which vanish; same idea in 2d.)

More generally, TFT's are special kinds of QFT's which contain a "topological subsector" of correlators whose correlation functions reduce to purely zero mode calculations. Since they reduce to zero mode calculations, we can get the *exact* answer (instead of merely some asymptotic expansion) for the correlation function merely by doing a bit of math.

Put more simply still, TFT's allow us to reduce *a priori* computationally difficult physics problems to easy math problems.

"Half-twisted" (0,2) model:

$$g_{i\bar{\jmath}} \overline{\partial} \phi^i \partial \phi^{\bar{\jmath}} \; + \; i h_{a\bar{b}} \lambda_-^{\bar{b}} D_z \lambda_-^a \; + \; i g_{i\bar{\jmath}} \psi_+^{\bar{\jmath}} D_{\overline{z}} \psi_+^i \; + \; F_{i\bar{\jmath}a\bar{b}} \psi_+^i \psi_+^{\bar{\jmath}} \lambda_-^a \lambda_-^{\bar{b}}$$

$$\begin{array}{cc} \lambda_-^a \; \in \; \Gamma(\phi^* \overline{\mathcal{E}}) & \psi_+^i \; \in \; \Gamma(K \otimes \phi^* T^{1,0} X) \\ \lambda_-^{\bar{b}} \; \in \; \Gamma(K \otimes \phi^* \overline{\mathcal{E}}) & \psi_+^{\bar{\imath}} \; \in \; \Gamma((\phi^* T^{1,0} X)^\vee) \end{array}$$

RR states (Q cohomology):

$$\mathcal{O} \; \sim \; b_{\bar{\imath}_1 \cdots \bar{\imath}_n a_1 \cdots a_p} \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_n} \lambda_-^{a_1} \cdots \lambda_-^{a_p} \; \leftrightarrow \; H^n(X, \Lambda^p \mathcal{E}^\vee)$$

When $\mathcal{E} = TX$, reduces to the A model above, since $H^{p,q}(X) = H^q(X, \Lambda^q (TX)^\vee)$.

Symmetry properties of states

A model:

$$H^{p,q}(X) \;\cong\; H^{n-p,n-q}(X)^* \text{ for compact } n\text{-dim'l } X$$

(0,2) model:

$$H^q(X, \Lambda^p \mathcal{E}^\vee) \;\cong\; H^{n-q}(X, (\Lambda^{r-p}\mathcal{E}^\vee) \otimes (\Lambda^{top}\mathcal{E} \otimes K_X))^*$$

for compact n-dim'l X, rank r $\mathcal{E}$

We'll assume $\Lambda^{top}\mathcal{E}^\vee \cong K_X$, in add'n to anomaly cancellation $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$

- recovers symmetry property $H^q(X, \Lambda^p\mathcal{E}^\vee) \cong H^{n-q}(X, \Lambda^{r-p}\mathcal{E}^\vee)$
- makes path integral measure well-defined
- essential for correlation functions
- in CY compactification, guarantees a left-moving $U(1)$ that is essential for spacetime gauge symmetry

Anomaly cancellation

We just outlined why we'll assume $\Lambda^{top}\mathcal{E}^\vee \cong K_X$.

We'll also assume $\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$.

This is the "anomaly cancellation" condition arising from the Green-Schwarz mechanism

$$dH \;=\; \text{tr}\; F \wedge F \;-\; \text{tr}\; R \wedge R$$

This condition also manifests itself in the worldsheet theory, and can be derived (as we'll see later) for massive 2D QFT's w/ non-CY targets.

Why work with the A model, or this “half-twisted” theory? Why not work directly in physical untwisted theories?

Reason: the twisted theories give same answer, with less work.

Consider 3-point functions.

A model:
It’s an old story that

$$< \psi\psi\phi >_{phys,II} = < \psi\psi\psi >_A$$

b/c the spectral flow operator encoding $\phi \leftrightarrow \psi$ is equivalent to twisting the theory.

Why is including the spectral flow operator equivalent to twisting?

Very briefly, one twists by adding $\int (1/2)\omega \overline{\psi} \psi = \int (1/2) \omega J$ to action. If bosonize $J \sim \partial \phi$, then the term $\sim \int R \phi$. By concentrating curvature at points, so $R \sim \delta^2(z - z_0)$, we see that twisting $\sim$ inserting $\exp(\phi) \sim$ spectral flow.

3-point functions, cont'd
(0,2) model:
No longer have left-moving $\mathcal{N} = 2$ susy, but do have a left-moving $U(1)$ that becomes $U(1)_R$ on the (2,2) locus, and is crucial for gauge properties.

Ex: $\mathcal{E}$ is rank 3, breaking E_8 to E_6. E_6 is built from $SO(10) \times U(1)$.

$$\overline{\mathbf{27}} \; = \; \mathbf{10}_{-1} \oplus \overline{\mathbf{16}}_{1/2} \oplus \mathbf{1}_2$$

so $\overline{\mathbf{27}}^3$ calculated as $< \psi_{\overline{\mathbf{16}}} \psi_{\overline{\mathbf{16}}} \phi_{\mathbf{10}} >_{phys,het}$
For same reasons as for the A model,

$$< \psi_{\overline{\mathbf{16}}} \psi_{\overline{\mathbf{16}}} \phi_{\mathbf{10}} >_{phys,het} = < \psi \psi \psi >_{half-twisted}$$

Classical correlation functions

A model:
For X compact, n-dim'l, have n χ^i zero modes and n $\chi^{\bar{\imath}}$ zero modes, plus bosonic zero modes $\sim X$, so

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > = \int_X H^{p_1,q_1}(X) \wedge \cdots \wedge H^{p_m,q_m}(X)$$

Selection rule from left-, right-moving $U(1)$'s:
$\sum_i p_i = \sum_i q_i = n$. Thus

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \sim \int_X (\text{top-form})$$

Classical correlation functions

(0,2) model:
Here we have n $\psi_+^{\bar{\imath}}$ zero modes and r λ^a zero modes, so

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > = \int_X H^{q_1}(X, \Lambda^{p_1} \mathcal{E}^\vee) \wedge \cdots \wedge H^{q_m}(X, \Lambda^{p_m} \mathcal{E}^\vee)$$

Selection rule from left-, right-moving $U(1)$'s: $\sum_i q_i = n$, $\sum_i p_i = r$. Thus

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \sim \int_X H^{top}(X, \Lambda^{top} \mathcal{E}^\vee)$$

When $\Lambda^{top} \mathcal{E}^\vee \cong K_X$, then the integrand is a top-form.

Next: worldsheet instantons

Worldsheet instantons

A model:

Here, moduli space of bosonic zero modes = moduli space of worldsheet instantons, $\mathcal{M}$.

We'll assume $\mathcal{M}$ is smooth, and review its compactification later.

Here again, correlation f'ns

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \sim \int_{\mathcal{M}} (\text{top form})$$

Worldsheet instantons

(0,2) model:
In addition to $\mathcal{M}$, the bundle $\mathcal{E}$ on X induces a bundle (of λ zero modes) $\mathcal{F}$ on $\mathcal{M}$:

$$\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$$

where $\alpha : \Sigma \times \mathcal{M} \rightarrow X$, and $\pi : \Sigma \times \mathcal{M} \rightarrow \mathcal{M}$.
On the (2,2) locus, where $\mathcal{E} = TX$, have $\mathcal{F} = T\mathcal{M}$ (fixed cpx structure on worldsheet)

When no excess zero modes ($R^1 \pi_* \alpha^* \mathcal{E} = 0 = R^1 \pi_* \alpha^* TX$),

$$\left. \begin{array}{c} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \end{array} \right\} \stackrel{GRR}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

Worldsheet instantons

(0,2) model, cont'd

Correlation functions look like

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \sim \int_{\mathcal{M}} H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{F}^\vee)$$

(no excess zero modes)

Classically, the integrand was a top-form b/c $\Lambda^{top}\mathcal{E}^\vee \cong K_X$.

Here, the integrand is a top form b/c (GRR) $\Lambda^{top}\mathcal{F}^\vee \cong K_{\mathcal{M}}$.

Cohomology on $X \mapsto$ cohomology on $\mathcal{M}$

A model:

Each element of $H^{p,q}(X)$ plus a point p on the worldsheet Σ define an element of $H^{p,q}(\mathcal{M})$,

by,

pullback along $\alpha|_{p \times \mathcal{M}}$, where $\alpha : \Sigma \times \mathcal{M} \to X$.

Cohomology on $X \mapsto$ cohomology on $\mathcal{M}$

(0,2) model:
Each element of $H^q(X, \Lambda^p \mathcal{E}^\vee)$ plus point p on worldsheet Σ define an element of $H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee)$:

1. first pullback along $\alpha|_{p \times \mathcal{M}}$ to get an element of $H^q(\mathcal{M}, \Lambda^p (\alpha^* \mathcal{E})^\vee|_{p\times\mathcal{M}})$
2. next use map

$$\mathcal{F} \, (\equiv \pi_* \alpha^* \mathcal{E}) \longrightarrow \alpha^* \mathcal{E}|_{p\times\mathcal{M}}$$

to define map

$$\Lambda^p \left(\alpha^* \mathcal{E}\right)^\vee |_{p\times\mathcal{M}} \longrightarrow \Lambda^p \mathcal{F}^\vee$$

When $\mathcal{E} = TX$, this reduces to the A model map.

Excess zero modes

A model:
Use 4-fermi term $\int_\Sigma R_{i\bar{\jmath}k\bar{l}} \chi^i \chi^{\bar{\jmath}} \psi^k \psi^{\bar{l}}$.
For each cpx pair of ψ zero modes, bring down one copy of 4-fermi term above.
Result:

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \sim \int_{\mathcal{M}} H^{\sum p_i, \sum q_i}(\mathcal{M}) \wedge c_{top}(\text{Obs})$$

where

$$\begin{aligned} \text{Obs} &= \text{bundle over } \mathcal{M} \text{ defined by } \psi \text{ zero modes} \\ &= R^1 \pi_* \alpha^* TX \\ &= \text{“obstruction bundle”} \end{aligned}$$

Excess zero modes

A model, cont'd:

Selection rules: $\sum p_i = \sum q_i = \#\chi - \#\psi$ zero modes.

$$\begin{aligned} \#\psi \text{ zero modes} &= \text{rank Obs} \\ \#\chi \text{ zero modes} &= \dim \mathcal{M} \end{aligned}$$

$$\sum p_i \; + \; (\text{rank Obs}) \; = \; \sum q_i \; + \; (\text{rank Obs}) \; = \; \dim \mathcal{M}$$

$\Longrightarrow$ integrand is a top form

Excess zero modes

(0,2) model:

Assume rk $R^1\pi_*\alpha^*\mathcal{E} =$ rk $R^1\pi_*\alpha^*TX = n$.

Use 4-fermi term $\int_\Sigma F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$.

$$\begin{array}{ll} \psi_+^{\bar{\jmath}} \sim T\mathcal{M} = R^0\pi_*\alpha^*TX & \lambda_-^a \sim \mathcal{F} = R^0\pi_*\alpha^*\mathcal{E} \\ \psi_+^i \sim \text{Obs} = R^1\pi_*\alpha^*TX & \lambda_-^{\bar{b}} \sim \mathcal{F}_1 \equiv R^1\pi_*\alpha^*\mathcal{E} \end{array}$$

Each 4-fermi $\sim H^1(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (\text{Obs})^\vee)$.

$$< \mathcal{O}_1 \cdots \mathcal{O}_m > \;\sim\; \int_{\mathcal{M}} H^{\sum q_i}\left(\mathcal{M}, \Lambda^{\sum p_i}\mathcal{F}^\vee\right) \wedge$$

$$H^n\left(\mathcal{M}, \Lambda^n\mathcal{F}^\vee \otimes \Lambda^n\mathcal{F}_1 \otimes \Lambda^n(\text{Obs})^\vee\right)$$

Excess zero modes

(0,2) model, cont'd:
Selection rules:

$$\begin{aligned} \sum q_i + n &= \text{dim } \mathcal{M} \\ \sum p_i + n &= \text{rank } \mathcal{F} \end{aligned}$$

and by assumption, rk $\mathcal{F}_1 =$ rk Obs $= n$.

$$\left. \begin{aligned} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \end{aligned} \right\}$$

$$\stackrel{GRR}{\Longrightarrow} \Lambda^{top} \mathcal{F}^\vee \otimes \Lambda^{top} \mathcal{F}_1 \otimes \Lambda^{top} (\text{Obs})^\vee \cong K_{\mathcal{M}}$$

Once again, integrand is a top-form.

We just presented an ansatz for interpreting 4-fermi terms in (0,2) models, and observed that GRR $\Rightarrow$ integrand a top-form, as needed.

But why does it reduce to (2,2) case when $\mathcal{E} = TX$?

Answer: Atiyah classes

Atiyah classes

Consider the curvature of a connection on a hol' bundle $\mathcal{E}$ on X:

$$F_{i\bar{\jmath}a\bar{b}}$$

Bianchi: $\overline{\partial} F = 0$, so $[F] \in H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right)$.

Since $\mathrm{ch}_r(\mathcal{E}) \propto$ tr $F \wedge \cdots \wedge F$ (r times), the Chern classes of $\mathcal{E}$ are encoded in

$$\begin{aligned} H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \wedge \cdots \wedge H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \\ = \; H^r\left(X, \Omega^r_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \end{aligned}$$

Let's specialize for a moment to $\mathcal{E} = TX$, so $\mathcal{F} = T\mathcal{M}$.
Each (0,2) 4-fermi term generates a factor of

$$H^1\left(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (\text{Obs})^\vee\right) \stackrel{\mathcal{E}=TX}{=} H^1\left(\mathcal{M}, \Omega^1_{\mathcal{M}} \otimes (\text{Obs})^\vee \otimes \text{Obs}\right)$$

$\rightarrow$ same gp that contains the Atiyah class of Obs bundle

Bringing down ($n =$ rk Obs) factors generates

$$H^n\left(\mathcal{M}, \Omega^n_{\mathcal{M}} \otimes \Lambda^{top}(\text{Obs})^\vee \otimes \Lambda^{top}\text{Obs}\right)$$

which contains $c_{top}(\text{Obs})$.

Thus, our (0,2) ansatz generalizes (2,2) obstruction bdles

Compactifications of moduli spaces

In order to make sense of expressions such as

$$\int_{\mathcal{M}} (\text{top form})$$

we need $\mathcal{M}$ to be compact.

Problem: spaces of honest holomorphic maps *not* compact

Ex: Degree 1 maps $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ = group manifold of $SL(2, \mathbf{C})$

How to solve? Regularize the 2d QFT: compactify $\mathcal{M}$.

Compactifications of moduli spaces

We just argued that to make sense of formal calculations, must compactify $\mathcal{M}$, *i.e.* add some measure-zero pieces that make $\mathcal{M}$ compact.

Furthermore, in the (0,2) case, need to extend $\mathcal{F}$, $\mathcal{F}_1$ over the compactification, in a way consistent with symmetries.

How to compactify? One way (Morrison-Plesser; Givental) uses gauged linear sigma models. We'll follow their lead.

Gauged linear sigma models

(2,2) case:
A chiral superfield in 2d contains

ϕ	cpx boson
ψ_+, ψ_-	cpx fermions
F	auxiliary field

Ex: A GLSM describes $\mathbf{P}^{N-1}$ as, N chiral superfields each of charge 1 w.r.t. gauged $U(1)$.

D-terms: $\sum |\phi_i|^2 = r \Longrightarrow \phi$'s span S^{2N-1}
Gauge-invariants: $S^{2N-1}/U(1) = \mathbf{P}^{N-1}$

Gauged linear sigma models

Can use GLSM's to describe more general toric varieties; look like, some chiral superfields + gauged $U(1)$'s
Can describe CY's by adding superpotential; zero locus of bosonic potential = CY

1. massive 2D QFT's, not CFT's
2. linear kinetic terms make analysis of some aspects of QFT easier than in a NLσM

Today I'll only consider (mostly massive) theories w/ toric targets.

(0,2) GLSM's

(0,2) chiral superfield Φ	(0,2) fermi superfield Λ
ϕ (cpx boson)	ψ_- (cpx fermion)
ψ_+ (cpx fermion)	F (aux field)

Together, form (2,2) chiral multiplet.

The fermi superfields have an important quirk: Although $\overline{D}_+ \Phi = 0$ for Φ chiral, can permit $\overline{D}_+ \Lambda = E$ for nonzero E obeying $\overline{D}_+ E = 0$. This constrains the superpotential; details soon....

Can describe a toric variety target as a collection of (0,2) chiral superfields with some gauged $U(1)$'s.
The (left-moving) fermi multiplets define bundles.

Bundles on toric varieties

Ex: Reducible case, $\mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a)$.

In GLSM have fermi superfields Λ^a w/ charges $\vec{n}_a$ under some $U(1)$'s

Ex: Kernel,

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{F_a^i} \oplus_i \mathcal{O}(\vec{m}_i) \longrightarrow 0$$

Have fermi superfields Λ^a as above, plus chiral superfields p_i of charges $\vec{m}_i$, plus superpotential term $p_i F_a^i(\phi)$.

Resulting Yukawa couplings $\psi_+ i F_a^i \lambda^a$ give mass to any λ not in ker F, hence, $\mathcal{E} =$ ker F.

Bundles on toric varieties

Ex: Cokernel,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \stackrel{E_a^i}{\longrightarrow} \oplus_a \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{E} \longrightarrow 0$$

Have fermi superfields Λ^a w/ charges $\vec{n}_a$ as above, plus k neutral chiral superfields Σ_i, where $\overline{D}_+ \Lambda^a = \Sigma_i E_a^i$.

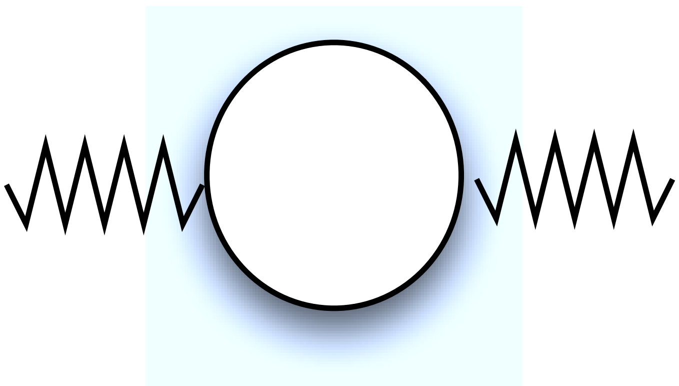
Ex: Monad,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \stackrel{E_a^{i'}}{\longrightarrow} \oplus_a \mathcal{O}(\vec{n}_a) \stackrel{F_a^i}{\longrightarrow} \oplus_i \mathcal{O}(\vec{m}_i) \longrightarrow 0$$

Have $\Sigma_{i'}$, Λ^a, p_i as above, w/ superpotential and susy transformation.

Anomaly cancellation

Let $\vec{n}_a$ denote charges of left-moving fermions, $\vec{q}_i$ denote charges of right-moving fermions.



Anom' cancellation implies

$$\sum_a n_a^t n_a^s = \sum_i q_i^t q_i^s$$

for each s, t

This implies, but is slightly stronger than, $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$.

We'll also assume $\sum_a n_a^t = \sum_i q_i^t$ for each t, which implies, but is slightly stronger than, $c_1(\mathcal{E}) = c_1(TX)$.

Linear sigma model compactifications

Basic idea:

1. expand fields in a basis of zero modes; if x_i has charges $\vec{q}_i$, then zero modes are $x_i \in \Gamma(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d}))$
2. coefficients are homogeneous coordinates on $\mathcal{M}$
3. build $\mathcal{M}$ like a Higgs moduli space (symplectic quotient)
   (a) exclude those zero modes that force the x_i to lie in excluded set for all points on worldsheet
   (b) the zero modes of x_i have same $U(1)$ charges as the original x_i

Linear sigma model compactifications

Ex: $\mathbf{P}^{N-1}$

Has N chiral superfields $x_1, \cdots, x_N$, one gauged $U(1)$, each x_i has charge 1.

The *gauge* instantons of the GLSM become the *worldsheet* instantons of the NLσM.

Moduli space of degree d maps here:

$$\begin{aligned} x_i &\in \Gamma(\mathcal{O}(1 \cdot d)) \\ &= x_{i0}u^d + x_{i1}u^{d-1}v + \cdots + x_{id}v^d \end{aligned}$$

where u, v are homogeneous coordinates on worldsheet ($\mathbf{P}^1$).

Linear sigma model compactifications

Ex, cont'd

The (x_{ij}) are homogeneous coord's on $\mathcal{M}$. Omit point where all $x_i \equiv 0$. The (x_{ij}) have same $U(1)$ charges as x_i for each x_i, thus

$$\mathcal{M} \; = \; \mathbf{P}^{N(d+1)-1}$$

Induced bundles

The same ideas allow us to induce bundles on LSM moduli spaces.

Just as worldsheet fields define line bundles on target, expand in zero modes, and coefficients define line bundles on $\mathcal{M}$.

Next: examples....

Induced bundles

Ex: completely reducible bundles, $\mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a)$

The left-moving fermions are completely free (mod action of the gauge group).

Expand each fermion in zero modes, take coeff's to define line bundles on $\mathcal{M}$.
Here, λ_-^a has charges $\vec{n}_a$. Expand

$$\lambda_-^a \;=\; \lambda_-^{a0} u^{\vec{n}_a \cdot \vec{d}+1} \;+\; \lambda_-^{a1} u^{\vec{n}_a \cdot \vec{d}} v \;+\; \cdots$$

Each $\lambda_-^{ai} \sim \mathcal{O}(\vec{n}_a)$ on $\mathcal{M}$. Thus,

$$\mathcal{F} \;=\; \oplus_a H^0 \left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Induced bundles

Ex (completely reducible bundles), cont'd

Similarly,

$$\mathcal{F}_1 \;=\; \oplus_a H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Induced bundles

Ex: Cokernel

$$0 \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow \oplus_a \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{E} \longrightarrow 0$$

In add'n to fermi superfields Λ^a for the $\mathcal{O}(\vec{n}_a)$, recall have chiral superfields Σ_j for the $\mathcal{O}$'s. As before, expand fields in basis of zero modes and interpret coefficients as line bundles on $\mathcal{M}$.

$$\begin{array}{ccl} 0 & \rightarrow & \oplus_1^m H^0\left(\mathcal{O}(0\cdot\vec{d})\right)\otimes\mathcal{O} \;\rightarrow\; \oplus_a H^0\left(\mathcal{O}(\vec{n}_a\cdot\vec{d})\right)\otimes\mathcal{O}(\vec{n}_a) \\ & & \rightarrow\; \mathcal{F} \\ & \rightarrow & \oplus_1^m H^1\left(\mathcal{O}(0\cdot\vec{d})\right)\otimes\mathcal{O} \;\rightarrow\; \oplus_a H^1\left(\mathcal{O}(\vec{n}_a\cdot\vec{d})\right)\otimes\mathcal{O}(\vec{n}_a) \\ & & \rightarrow\; \mathcal{F}_1 \;\rightarrow\; 0 \end{array}$$

Induced bundles

Ex: Cokernel, cont'd

Since $H^1(\mathbf{P}^1, \mathcal{O}) = 0$, this simplifies to

$$0 \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow \oplus_a H^0\left(\mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_a H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Check (2,2) locus

The tangent bundle of a (cpt, smooth) toric variety X can be expressed in the form

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0$$

where the $\vec{q}_i$ are the charges of the chiral superfields.

Applying previous ansatz,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_i H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i)$$

but this $\mathcal{F}$ is automatically $T\mathcal{M}$ for $\mathcal{M}$ a LSM moduli space, exactly as desired.

Check (2,2) locus

Also, $\mathcal{F}_1 =$obstruction bundle.

Check:

$$c_{top}(\mathcal{F}_1) \;=\; \prod_{\vec{n}_a\cdot\vec{d}<0} c_1(\mathcal{O}(\vec{n}_a))^{-\vec{n}_a\cdot d-1}$$

Similar ideas hold for other bundles appearing in (0,2) GLSM's.

In all cases: so long as the original gauge bundle satisfied GLSM anomaly cancellation, the induced bundles $\mathcal{F}$, $\mathcal{F}_1$ have the desired symmetry properties.

Also, if a given bundle does not satisfy GLSM anomaly cancellation, then the induced bundles $\mathcal{F}$, $\mathcal{F}_1$ often won't have the desired symmetry properties.

Presentation-dependence

Here's an example of what can happen with GLSM anomaly cancellation.
Consider the tangent bundle T of $\mathbf{P}^1 \times \mathbf{P}^1$. This has (at least) 3 presentations:

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow T \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,2) \longrightarrow T \longrightarrow 0$$
$$T \cong \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$$

The same bundle, but only the first presentation satisfies GLSM anomaly cancellation. Next: compute $\mathcal{F}$....

Presentation-dependence

LSM $\mathcal{M} = \mathbf{P}^{2d_1+1} \times \mathbf{P}^{2d_2+1}$
Induced bundles:

$$\mathcal{F} \cong T\mathbf{P}^{2d_1+1} \times \mathbf{P}^{2d_2+1}$$
$$\mathcal{F} \cong \pi_1^* T\mathbf{P}^{2d_1+1} \oplus \oplus_1^{2d_2+1} \mathcal{O}(0,2)$$
$$\mathcal{F} \cong \oplus_1^{2d_1+1} \mathcal{O}(2,0) \oplus \oplus_1^{2d_2+1} \mathcal{O}(0,2)$$

These are isomorphic on the interior of $\mathcal{M}$, on the honest maps, but differ over the compactification.

Only in the first case (which was the only one to satisfy GLSM anomaly cancellation) is $\Lambda^{top}\mathcal{F}^\vee \cong K_\mathcal{M}$.

Adams-Basu-Sethi prediction

Adams-Basu-Sethi studied a massive 2d theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with a bundle given by a deformation of the tangent bundle.

From analysis of duality in the corresponding massive (0,2) gauged linear sigma model, they made some conjectures for correlation f'ns, which they expressed in terms of a "heterotic quantum cohomology ring."

Chiral rings

The idea of a chiral ring should be familiar from 4d susy gauge theories, *e.g.* Cachazo-Douglas-Seiberg-Witten.

4d $N = 1$ pure $SU(N)$ SYM	2d susy $\mathbf{CP}^{N-1}$ model
$S^N = \Lambda^{3N}$	$x^N = q$
$W = S\left(1 + \log(\Lambda^{3N}/S^N)\right)$	$W = \Sigma\left(1 + \log(\Lambda^N/\Sigma^N)\right)$
Konishi	Konishi
etc	*etc*

where for the $\mathbf{CP}^N$ model, the x is identified with a generator of $H^2(\mathbf{CP}^N, \mathbf{Z})$, and so the physical ring relation looks like a modification of the std cohomology ring $\mathbf{C}[x]/(x^N = 0)$, yielding “quantum cohomology” ring $\mathbf{C}[x]/(x^N = q)$.

Quantum cohomology

More concretely, the quantum cohomology ring of $\mathbf{CP}^N$ tells us that correlation functions are:

$$< x^k > = \begin{cases} q^m & \text{if } k = mN + N - 1 \\ 0 & \text{else} \end{cases}$$

Ordinarily use (2,2) worldsheet susy to argue for existence of a quantum cohomology ring.

- Adams-Basu-Sethi conjectured they might exist for (0,2)
- ES-Katz checked correlation f'ns, found ring structure
- Adams-Distler-Ernebjerg found gen'l argument for (0,2) ring structure

Adams-Basu-Sethi prediction

Adams-Basu-Sethi studied a massive 2d theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with a bundle given by a deformation of the tangent bundle, with a deformation specified by two parameters ϵ_1, ϵ_2.

From analysis of duality in the corresponding massive (0,2) gauged linear sigma model, they conjectured that the "quantum cohomology ring" should be a deformation of the usual ring:

$$\begin{aligned} \tilde{X}^2 &= \exp(it_2) \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= \exp(it_1) \end{aligned}$$

where the t_i are Kähler parameters describing the sizes of the $\mathbf{P}^1$'s, and X, $\tilde{X}$ are the two generators.

Adams-Basu-Sethi prediction

Conjectured relations:

$$\begin{aligned} \tilde{X}^2 &= \exp(it_2) \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= \exp(it_1) \end{aligned}$$

What do those ring relations really mean? For ex:

$$\begin{aligned} < \tilde{X}^4 > &= < 1 > \exp(2it_2) \; = \; 0 \\ < X \tilde{X}^3 > &= < (X\tilde{X}) \tilde{X}^2 > \\ &= < X \tilde{X} > \exp(it_2) \; = \; \exp(it_2) \\ < X^2 \tilde{X}^2 > &= < X^2 > \exp(it_2) \; = \; (\epsilon_1 - \epsilon_2) \exp(it_2) \\ < X^3 \tilde{X} > &= \exp(it_1) \; + \; (\epsilon_1 - \epsilon_2)^2 \exp(it_2) \\ < X^4 > &= 2(\epsilon_1 - \epsilon_2) \exp(it_1) \; + \; (\epsilon_1 - \epsilon_2)^3 \exp(it_2) \end{aligned}$$

Adams-Basu-Sethi prediction

To be brief, using exactly the methods described so far (compute $\mathcal{M}$, $\mathcal{F}$, compute induced sheaf cohomology on $\mathcal{M}$ in terms of Cech reps on toric cover, calculate $\wedge$'s on $\mathcal{M}$ & integrate), we precisely reproduced the results above.

(We calculated 4-pt interactions, and a grad student wrote a computer program to calculate higher-pt interactions.)

(0,2) B model analogue

So far I've outlined the (0,2) analogue of the A model. What about the B model?

Recall they differ by the choice of left twist:

A model	B model
$\psi_-^i \in \Gamma((\phi^* T^{0,1} X)^\vee)$	$\psi_-^i \in \Gamma(K \otimes (\phi^* T^{0,1} X)^\vee)$
$\psi_-^{\overline{\imath}} \in \Gamma(K \otimes (\phi^* T^{0,1} X)^\vee)$	$\psi_-^{\overline{\imath}} \in \Gamma((\phi^* T^{0,1} X)^\vee)$

One can define analogous (0,2) versions:

A analogue	B analogue
$\lambda_-^a \in \Gamma((\phi^* \overline{\mathcal{E}})^\vee)$	$\lambda_-^a \in \Gamma(K \otimes (\phi^* \overline{\mathcal{E}})^\vee)$
$\lambda_-^{\overline{a}} \in \Gamma(K \otimes \phi^* \overline{\mathcal{E}})$	$\lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$

(0,2) B model analogue

A analogue	B analogue
$\lambda_-^a \in \Gamma((\phi^* \overline{\mathcal{E}})^\vee)$	$\lambda_-^a \in \Gamma(K \otimes (\phi^* \overline{\mathcal{E}})^\vee)$
$\lambda_-^{\overline{a}} \in \Gamma(K \otimes \phi^* \overline{\mathcal{E}})$	$\lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$

Note that in the (0,2) version, we can go A $\leftrightarrow$ B by switching $\mathcal{E} \leftrightarrow \mathcal{E}^\vee$.

So, once you know the (0,2) analogue of the A model, you also know the (0,2) analogue of the B model – the same model generalizes both simultaneously.

Consistency conditions on (2,2) locus

Recall our (0,2) A model analogue had the following constraint on the gauge bundle $\mathcal{E}$:

$$\Lambda^{top} \mathcal{E}^\vee \; \cong \; K_X$$

To get the B model analogue, we replace $\mathcal{E}$ with $\mathcal{E}^\vee$, and so have another constraint:

$$\Lambda^{top} \mathcal{E} \; \cong \; K_X$$

Put together, these constraints imply

$$K_X^\vee \; \cong \; K_X \; \Longrightarrow \; K_X^2 \; \cong \; \mathcal{O}$$

Consistency conditions on (2,2) locus

We're used to saying the closed string B model is well-defined only on Calabi-Yau's ($K_X \cong \mathcal{O}$), but we just derived instead the condition that $K_X^2 \cong \mathcal{O}$.

And, in fact, it's an obscure fact that consistency of the closed string B model merely requires $K_X^2 \cong \mathcal{O}$, not actually $K_X \cong \mathcal{O}$.

Consistency conditions on (2,2) locus

- Implicit in old expressions for B model correlation f'ns, Kodaira-Spencer.
- Loop calculation yields ambiguous result, as it cannot sense torsion, and $K_X^2 \cong \mathcal{O} \Rightarrow c_1$ torsion
- Careful analysis of anomaly cancellation yields the desired result
- Check: Serre duality correctly maps massless spectrum into itself
- Exs: Enriques surfaces, hyperelliptic surfaces
- Open string B model requires stronger condition: $K \cong \mathcal{O}$

Summary

- review A model TFT, half-twisted (0,2) TFT
- review correlation f'n computations in A model, describe analogue for (0,2) models
  - formal structure similar; (0,2) generalizes A model
  - compactification issues; not only $\mathcal{M}$, but bundles on $\mathcal{M}$
- apply GLSM's; not only naturally compactify $\mathcal{M}$, but also naturally extend the bundles
- Adams-Basu-Sethi prediction
- Analogue for B model
- Consistency conditions in closed string B model

Thank you for your time!