Comments on Twistor Strings: Old and New

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October 26, 2006

Why are twistors interesting?

- Twistor string theory
  - A weak coupling version of the AdS/CFT duality (Witten)
- Calculation of gauge theory amplitudes
  - Direct calculation difficult due to large numbers of Feynman diagrams
  - Twistor techniques led to
    - A formula for tree level S-matrix in QCD ($\mathcal{N} = 4$ YM $\sim$ QCD at tree-level) (Witten; Spradlin, Roiban, Volovich, ...)
    - Many one-loop amplitudes (Number of different groups)
    - New recursion rules which facilitate calculation (Cachazo, Svrček, Witten; Britto, Cachazo, Feng, Witten; + ...)

Plan of the talk

- MHV amplitudes for gauge theory
- (Super)twistor space
- Twistor version of MHV and generalization
- Twistor strings (Witten, Berkovits)
- Graviton MHV amplitudes, hints of $\mathcal{N} = 8$ supergravity
- New twistor strings
- More graviton amplitudes, do we have $\mathcal{N} = 8$ supergravity?

The MHV amplitudes

(Maximally Helicity Violating amplitudes)

Gluons are massless, $p_\mu \;\Rightarrow\; p^2 = 0 \;\Rightarrow p_\mu$ is a null vector.

$$p^A_{\ \dot{A}} = (\sigma^\mu)^A_{\ \dot{A}}\, p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \qquad = \; \pi^A \bar{\pi}_{\dot{A}}$$

$(\pi,\ e^{i\theta}\pi) \longrightarrow$ same p_μ, $\qquad p_\mu$ is real $\Rightarrow \bar{\pi}_{\dot{A}} = (\pi^A)^*$

$$\pi = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix}, \qquad \bar{\pi} = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 + ip_2 \\ p_0 - p_3 \end{pmatrix}$$

For every momentum for a massless particle $\longrightarrow$ a spinor momentum π.

The MHV amplitudes (cont'd.)

- Lorentz transformation

$$\pi^A \to \pi'^A = (g\pi)^A = g^A_B \, \pi^B, \qquad g \in SL(2, \mathbf{C})$$

- Lorentz-invariant scalar product

$$\langle 12 \rangle = \pi_1 \cdot \pi_2 = \epsilon_{AB} \, \pi_1^A \, \pi_2^B$$

- Gluon helicity

$$\epsilon_\mu = \epsilon_{\dot{A}A} = \begin{cases} \bar{\pi}_{\dot{A}} \lambda_A / \pi \cdot \lambda & +1 \text{ helicity} \\ \\ \bar{\lambda}_{\dot{A}} \pi_A / \bar{\pi} \cdot \bar{\lambda} & -1 \text{ helicity} \end{cases}$$

Write amplitudes in terms of these invariants

The MHV amplitudes (cont'd.)

Results obtained in 1986 by Parke and Taylor, proved by Berends and Giele

$$\begin{aligned}
\mathcal{A}(1^{a_1}_{+}, 2^{a_2}_{+}, 3^{a_3}_{+}, \cdots, n^{a_n}_{+}) &= 0 \\
\mathcal{A}(1^{a_1}_{-}, 2^{a_2}_{+}, 3^{a_3}_{+}, \cdots, n^{a_n}_{+}) &= 0 \\
\mathcal{A}(1^{a_1}_{-}, 2^{a_2}_{-}, 3^{a_3}_{+}, \cdots, n^{a_n}_{+}) &= i g^{n-2} (2\pi)^4 \delta(p_1 + ... + p_n)\, \mathcal{M} \\
&\qquad + \text{noncyclic permutations}
\end{aligned}$$

$$\mathcal{M}(1^{a_1}_{-}, 2^{a_2}_{-}, 3^{a_3}_{+}, \cdots, n^{a_n}_{+}) = \langle 12 \rangle^4 \, \frac{\mathrm{Tr}(t^{a_1} t^{a_2} \cdots t^{a_n})}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1\; n \rangle \langle n1 \rangle}$$

We will rewrite this in three steps

The first step: The Dirac determinant

$$\begin{aligned}\mathrm{Tr}\log D_{\bar z} &= \mathrm{Tr}\log(\partial_{\bar z} + A_{\bar z})\\ &= \mathrm{Tr}\log\left(1 + \tfrac{1}{\partial_{\bar z}} A_{\bar z}\right) \quad + \ \text{constant}\end{aligned}$$

$$\mathrm{Tr}\log D_{\bar z} = \sum_n \int \frac{d^2x_1}{\pi}\frac{d^2x_2}{\pi}\cdots \frac{(-1)^{n+1}}{n}\frac{\mathrm{Tr}[A_{\bar z}(1)\cdots A_{\bar z}(n)]}{z_{12}\, z_{23}\cdots z_{n-1n}\, z_{n1}}$$

$$\left(\frac{1}{\partial_{\bar z}}\right)_{12} = \frac{1}{\pi(z_1 - z_2)} = \frac{1}{\pi\, z_{12}}$$

z's $\sim$ local coordinates on $\mathbf{CP}^1$.

$$\mathbf{CP}^1 = \{u^a,\ a = 1,2,\ |\ u^a \sim \rho u^a\}, \qquad u^a = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad \rho \neq 0$$

The Dirac determinant (cont'd.)

$z = \beta/\alpha$ on coordinate patch with $\alpha \neq 0$

$$z_1 - z_2 = \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} = \frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\alpha_1\alpha_2}$$

$$= \frac{\epsilon_{ab} u_1^a u_2^b}{\alpha_1\alpha_2} = \frac{u_1 \cdot u_2}{\alpha_1\alpha_2}$$

Define $\alpha^2 A_{\bar{z}} = \bar{\mathcal{A}}$

$$\mathrm{Tr}\log D_{\bar{z}} = -\sum \frac{1}{n} \int \frac{\mathrm{Tr}[\bar{\mathcal{A}}(1)\bar{\mathcal{A}}(2)\cdots\bar{\mathcal{A}}(n)]}{(u_1 \cdot u_2)(u_2 \cdot u_3)\cdots(u_n \cdot u_1)}$$

If $u^a \to \pi^A$, the denominators are right for YM amplitudes.

The second step: Helicity factors

- Lorentz generator

$$J_{AB} = \frac{1}{2}\left(\pi_A \frac{\partial}{\partial \pi^B} + \pi_B \frac{\partial}{\partial \pi^A}\right), \qquad \pi_A = \epsilon_{AB}\pi^B$$

- Spin operator $S_\mu \sim \epsilon_{\mu\nu\alpha\beta} J^{\nu\alpha} p^\beta$, $\quad J^{\mu\nu} =$ Lorentz generator
  $\Rightarrow S_{A\dot{A}} = J^A_{\ B}\, \pi^B \bar{\pi}_{\dot{A}} = -p^A_{\ \dot{A}}\; s$
- Helicity

$$s = -\frac{1}{2}\pi^A \frac{\partial}{\partial \pi^A} = -\frac{1}{2}\{\text{degree of homogeneity in } \pi^A\}$$

Consistent with powers of π in amplitude

Helicity factors (cont'd.)

θ_A = Anticommuting spinor $\Rightarrow \int d^2\theta \; \theta_A \theta_B = \epsilon_{AB} \;\; \Rightarrow$

$$\int d^2\theta \; (\pi\theta)(\pi'\theta) = \int d^2\theta \; (\pi^A\theta_A)(\pi'^B\theta_B) = \pi \cdot \pi'$$

Need 4 such powers $\Rightarrow \mathcal{N} = 4$ superfield

$$\bar{A}^a(\pi, \bar{\pi}) = a^a_+ + \xi^\alpha \; a^a_\alpha + \frac{1}{2}\xi^\alpha \xi^\beta \; a^a_{\alpha\beta} + \frac{1}{3!}\xi^\alpha \xi^\beta \xi^\gamma \epsilon_{\alpha\beta\gamma\delta} \; \bar{a}^{a\delta} + \xi^1 \xi^2 \xi^3 \xi^4 \; a^a_-$$

$$\xi^\alpha = (\pi\theta)^\alpha = \pi^A \theta^\alpha_A, \qquad \alpha = 1, 2, 3, 4$$

a^a_+ = Positive helicity gluon, $\quad a^a_-$ = Negative helicity gluon

$a^a_\alpha, \; \bar{a}^{a\alpha}, \; a^a_{\alpha\beta}$ = Spin-$\frac{1}{2}$ and spin-zero particles

The MHV formula for $\mathcal{N} = 4$ SYM

Gauge potential for Dirac determinant

$$\bar{\mathcal{A}} = g\, t^a \bar{A}^a \exp(ip \cdot x)$$

$$\Gamma[\bar{\mathcal{A}}] = \frac{1}{g^2} \int d^8\theta d^4x \; \mathrm{Tr} \log D_{\bar{z}} \Big]_{u^a \to \pi^A}$$

MHV amplitude is

$$\mathcal{A}(1_-^{a_1}, 2_-^{a_2}, 3_+^{a_3}, \cdots, n_+^{a_n})$$
$$= i \left[\frac{\delta}{\delta a_-^{a_1}(p_1)} \frac{\delta}{\delta a_-^{a_2}(p_2)} \frac{\delta}{\delta a_+^{a_3}(p_3)} \cdots \frac{\delta}{\delta a_+^{a_n}(p_n)} \Gamma[a] \right]_{a=0}$$

(Nair, 1988)

An alternate representation

$$\exp(i\eta\cdot\xi) = 1 \ + \ i\eta\cdot\xi \ + \ \frac{1}{2!} i\eta\cdot\xi \, i\eta\cdot\xi \ + \ \frac{1}{3!} i\eta\cdot\xi \, i\eta\cdot\xi \, i\eta\cdot\xi \\ + \ \frac{1}{4!} i\eta\cdot\xi \, i\eta\cdot\xi \, i\eta\cdot\xi \, i\eta\cdot\xi$$

$\eta\cdot\xi = \eta_\alpha \xi^\alpha$, state of particle $= |\pi, \eta\rangle$

$$\bar{\mathcal{A}} = g \, t^a a^a \exp(ip\cdot x + i\eta\cdot\xi)$$

Amplitudes $\sim$ coefficient of a^n in $\Gamma[\bar{\mathcal{A}}]$

For 1 and 2 of negative helicity, choose the coefficient of

$$\eta_{11}\eta_{21}\eta_{31}\eta_{41} \ \eta_{12}\eta_{22}\eta_{32}\eta_{42} \sim \prod_\alpha \eta_{\alpha 1} \prod_\beta \eta_{\beta 2}$$

(Super)twistor space

- Twistor

$$Z^\alpha = (W_{\dot{A}}, U^A), \qquad Z^\alpha \sim \lambda Z^\alpha, \qquad \lambda \neq 0 \implies \mathbf{CP}^3$$

- A holomorphic line in twistor space

$$\begin{array}{ccc} \mathbf{CP}^1 & \rightarrow & \mathbf{CP}^3 \\ u^a & & Z^\alpha \end{array}$$

$$W_{\dot{A}} = x_{\dot{A}A}\, u^A, \quad U^A = u^A \qquad \left[U^A = b^A_b u^b = u^A \text{ by } SL(2, \mathbf{C}) \right)$$

- Local complex coordinates on S^4

$$x_{\dot{A}A} = \begin{pmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{pmatrix} = x_4 + ix_i\sigma^i$$

(Super)twistor space (cont'd.)

$W_{\dot{A}}$= local complex coordinates on spacetime

- Moduli space of lines

  Moduli $\sim x_{\dot{A}A}$

  Spacetime $\sim$ moduli space of lines in twistor space

- $\mathcal{N} = 4$ supertwistor

$$(Z^\alpha, \xi^\alpha) = ((W_{\dot{A}}, U^A),\ \xi^\alpha), \qquad Z^\alpha \sim \lambda Z^\alpha,\ \ \xi^\alpha \sim \lambda \xi^\alpha$$

$$\Longrightarrow \mathbf{CP}^{3|4} \qquad (\text{Calabi}-\text{Yau supermanifold})$$

- A holomorphic line in supertwistor space

$$W_{\dot{A}} = x_{\dot{A}A}\, u^A, \qquad U^A = b^A_b u^b = u^A, \qquad \xi^\alpha = \theta^\alpha_a\, u^a$$

The third step: Lines in twistor space (Witten)

$$\exp(ip\cdot x) = \exp\left(\frac{i}{2}\bar{\pi}^{\dot{A}} x_{\dot{A}A}\pi^{A}\right) = \exp\left(\frac{i}{2}\bar{\pi}^{\dot{A}} W_{\dot{A}}\right)\Bigg]_{u^A=\pi^A}$$

$W_{\dot{A}} = x_{\dot{A}A}u^{A}$. Regard $W_{\dot{A}}$ as a free variable,

$$\int d\sigma\, \delta\left(\frac{\pi^2}{\pi^1} - \frac{U^2}{U^1}\right) \exp\left(\frac{i}{2}\bar{\pi}^{\dot{A}}\pi^1 \frac{W_{\dot{A}}}{U^1}\right) = \exp\left(\frac{i}{2}\bar{\pi}^{\dot{A}} x_{\dot{A}A}\pi^{A}\right) = \exp(ip\cdot x)$$

setting $W_{\dot{A}} = x_{\dot{A}A}\, u^{A}$, $\;U^{A} = u^{A}$, $\;\sigma = u^2/u^1$, local coordinate on $\mathbf{CP}^1$

Lines in twistor space (cont'd.)

$$\mathcal{A} = i g^{n-2} \int d^4x d^8\theta \int d\sigma_1 \cdots d\sigma_n$$

$$\frac{\mathrm{Tr}(t^{a_1} \cdots t^{a_n})}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_n - \sigma_1)} \prod_i \delta\left(\frac{\pi_i^2}{\pi_i^1} - \frac{U^2(\sigma_i)}{U^1(\sigma_i)}\right)$$

$$\times \exp\left(\frac{i}{2} \bar{\pi}_i^{\dot{A}} \pi_i^1 \frac{W_{\dot{A}}(\sigma_i)}{U^1(\sigma_i)} + i \pi_i^1 \eta_{\alpha i} \frac{\xi^\alpha(\sigma_i)}{U^1(\sigma_i)}\right)$$

$$+ \text{ noncyclic permutations}$$

$$W_{\dot{A}} = x_{\dot{A}A}\, u^A, \qquad U^A = u^A, \qquad \xi^\alpha = \theta^\alpha_a\, u^a$$

Remark: Calculate with signature $(++--)$ (real twistors) and continue

Properties of the amplitude $\mathcal{A}$

- Holomorphic in the twistor variables Z^α, ξ^α, holomorphic in the variable σ or u^a.
- Invariant under $Z^\alpha \to \lambda Z^\alpha$, $\xi^\alpha \to \lambda \xi^\alpha$,
- Has support only on a curve of degree one in supertwistor space
- Integration over the moduli $x_{\dot{A}A}$, θ^α_A

One can obtain the amplitude by taking

1. Holomorphic map $\mathbf{CP}^1 \to \mathbf{CP}^{3|4}$, degree one
2. Pick n points $\sigma_1, \sigma_2, \cdots, \sigma_n$
3. Evaluate the integral in over σ's, the moduli of the chosen map

Generalization to non-MHV amplitudes

Use a holomorphic map of degree d where $d+1$ is the number of negative helicity gluons

$$W_{\dot{A}}(\sigma) = (u^1)^d \sum_0^d b_{\dot{A}k}\sigma^k, \qquad U^A(\sigma) = (u^1)^d \sum_0^d a_k^A \sigma^k$$

$$\xi^\alpha(\sigma) = (u^1)^d \sum_0^d \gamma_k^\alpha \sigma^k$$

Integration over moduli

$$d\mu = \frac{d^{2d+2}a \; d^{2d+2}b \; d^{4d+4}\gamma}{vol[GL(2,\mathbf{C})]}$$

Scale invariance + $SL(2,\mathbf{C}) \Rightarrow GL(2,\mathbf{C})$

(Explicit checks by Spradlin, Roiban, Volovich + others)

Justification: Twistor strings

1. (Witten) Topological B-model, target space $\mathbf{CP}^{3|4}$

Open strings which end on $D5$-branes, $\bar{\xi}^{\alpha} = 0, \Rightarrow \bar{\mathcal{A}}(Z, \bar{Z}, \xi)$

$$\mathcal{I} = \frac{1}{2} \int_Y \Omega \wedge \mathrm{Tr}(\bar{\mathcal{A}}\, \bar{\partial}\, \bar{\mathcal{A}} + \frac{2}{3} \bar{\mathcal{A}}^3)$$

$Y \subset \mathbf{CP}^{3|4}, \quad \bar{\xi} = 0$

$$\Omega = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta} Z^{\alpha} dZ^{\beta} dZ^{\gamma} dZ^{\delta}\ d\xi_1 d\xi_2 d\xi_3 d\xi_4$$
$$= \text{top}-\text{rank holomorphic form on } \mathbf{CP}^{3|4}$$

Equations of motion $\Rightarrow \bar{\partial}\bar{\mathcal{A}} + \ldots = 0$

Holomorphic fields on twistor space $\Rightarrow$ massless fields on spacetime (Penrose correspondence)

Twistor strings (cont'd.)

Effective action in spacetime

$$\mathcal{I} = \int \mathrm{Tr}\left[G^{AB} F_{AB} + \bar{\chi}^{A\alpha} D_{A\dot{A}} \chi_{\alpha}^{\dot{A}} + \cdots \right]$$

G^{AB} = self-dual field, helicity -1, $A \sim$ helicity $+1$

$D1$-branes (instantons) $\Rightarrow +\frac{1}{2}\int G^2 \epsilon$

Integrate this out $\Rightarrow \mathcal{N} = 4$ YM with $\epsilon \sim g^2$

$$\langle GA \rangle \sim 1, \qquad \langle AA \rangle \sim \epsilon, \qquad GAA\text{−vertex}$$

$(d+1)$ G's $\Rightarrow$ d ϵ's $\Rightarrow$ Instanton number = d

$d+1$ negative helicity gluons $\Rightarrow$ Holomorphic maps of degree d

Twistor strings (cont'd.)

2. Berkovits' string theory

- The action is

$$\mathcal{S} = \int Y(\bar{\partial} + \bar{A})Z \; + \; S_C$$

Z stands for $(W_A, U^{\dot{A}}, \xi^\alpha)$, similarly for Y.

- $\bar{A}$ is a $GL(1, \mathbf{C})$ gauge field,

$$Z \to \lambda Z, \quad Y \to \lambda^{-1} Y, \quad \bar{A} \to \bar{A} - \bar{\partial} \log \lambda$$

- On the sphere, there are monopole configurations for $\bar{A}$,

$$\bar{A} = \bar{A}_d + \bar{\partial}\Theta, \qquad [d\bar{A}] = \sum_d [d\Theta] \det(\bar{\partial})$$

Twistor strings (cont'd.)

- For genus zero, use $\mathbf{CP}^1$; there are zero modes for Z,

$$Z^\alpha = \sum_{\{a\}} a^\alpha_{a_1 a_2 \cdots a_d} \, u^{a_1} u^{a_2} \cdots u^{a_d} \; + \text{higher nonzero modes}$$

$$\xi^\alpha = \sum_{\{a\}} \gamma^\alpha_{a_1 a_2 \cdots a_d} \, u^{a_1} u^{a_2} \cdots u^{a_d} \; + \cdots$$

- Higher nonzero modes have $\bar{u}$'s with $N_u - N_{\bar{u}} = d$. They are not holomorphic lines.
- Correlators have the form $\sum_d \mathcal{C}_d M_d$,

$$M_d = \int \underbrace{[\det \bar{\partial}] \det(\bar{D}) e^{-S_C} e^{-\int Y(\bar{\partial}+\bar{A}_d)Z}}_{\Rightarrow\, C = D - N - 28 + C_C} \quad V_1 V_2 \ldots V_n$$

Twistor strings (cont'd.)

- The vertex operators for gauge fields are given by $V_\Phi = \int d\sigma\, \Phi(Z)\, J$.
- Φ is holomorphic of degree zero in Z. It is of degree -2 in the spinor momentum Π.
- J is current for S_C and

$$
\begin{aligned}
\Phi(\Pi,\eta) =& \delta\left[\Pi \cdot Z(\sigma)\right] \frac{Z(\sigma)\cdot A}{\Pi \cdot A} \\
& \times \exp\left(\frac{i}{2} \frac{\bar{\Pi}\cdot Z(\sigma)\, \Pi\cdot A}{Z(\sigma)\cdot A} + i \frac{\Pi \cdot A}{Z(\sigma)\cdot A} \eta \cdot \xi(\sigma) \right)
\end{aligned}
$$

$$
\Pi^\alpha = (0, \pi^A) = (0, 0, \pi^1, \pi^2), \qquad A_\alpha = (0,0,1,0)
$$

Twistor strings (cont'd.)

- Correlators for V_Φ are saturated by zero modes, integration over fields is now integration over moduli of zero modes (holomorphic curve) $\Longrightarrow$ previous expression
- There are other vertex operators which give (super)gravitons.
- The theory becomes the theory of $N = 4$ SYM coupled to $N = 4$ conformal supergravity.
- It would be interesting/useful to if one can decouple gravity. The conformal gravitons occur in loops, there is no dimensional parameter which can be tuned to eliminate them.
- Independently, we can ask, is there a similar story for gravitons of the Einstein theory?

Graviton amplitudes

- MHV amplitudes for gravitons has been calculated (Berends, Giele, Kuijf)

$$\mathcal{A} = \left(\frac{\kappa}{2}\right)^{n-2} \delta^{(4)}\big(\sum_i p_i\big)\, \mathcal{M}$$

$$\mathcal{M} = \langle 12\rangle^8 \left[\frac{[12][n-2\; n-1]}{\langle 1\; n-1\rangle} \frac{1}{N(n)} \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle ij\rangle\, F \right.$$
$$\left. + \mathcal{P}(2,3,...,n-2) \right]$$

where $N(n) = \prod_{i,j,i<j} \langle ij\rangle$, $\quad \kappa = \sqrt{32\pi G}$.
$\mathcal{P}(2,3,...,n-2)$ indicates permutations of the labels $2,3,...,n-2$.

Graviton amplitudes (cont'd.)

- The quantity F is

$$F = \begin{cases} \prod_{l=3}^{n-3} \bar{\pi}_{\dot{A}l}(p_{l+1} + p_{l+2} + \cdots + p_{n-1})^{\dot{A}}_{A}\pi^{A}_{n} & n \geq 6 \\ 1 & n = 5 \end{cases}$$

- $[ij]$ is the product $\epsilon^{\dot{A}\dot{B}}\bar{\pi}_{\dot{A}i}\bar{\pi}_{\dot{B}j}$. This is antiholomorphic.
- The 4-point amplitude is

$$\mathcal{M}(1_-, 2_-, 3_+, 4_+) = \langle 12\rangle^8 \left[\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right] \times \left[\frac{[2P4\rangle}{\langle 24\rangle}\frac{1}{\langle 13\rangle\langle 34\rangle\langle 41\rangle}\right]$$

$$\frac{[2P4\rangle}{\langle 24\rangle} = \frac{\bar{\pi}_{\dot{A}2}P^{\dot{A}}_{A}\pi^{A}_{4}}{\langle 24\rangle} = -\frac{[12]\langle 41\rangle}{\langle 24\rangle}, \qquad P = p_2 + p_3 + p_4$$

Graviton amplitudes (cont'd.)

- Introduce a set of fermion fields ϕ, χ,

$$\langle \phi(1)\chi(2)\rangle = \langle \phi_1 \chi_2 \rangle = \frac{1}{\langle 12 \rangle}$$

- We can use a Penrose contour integral

$$\oint_{C_4} \frac{\epsilon_{AB}\lambda^A d\lambda^B}{2\pi i} \frac{1}{\langle 4\lambda\rangle\langle\lambda 1\rangle} f(\lambda) = \frac{1}{\langle 41\rangle} f(4)$$

- This leads to

$$\frac{[2P4\rangle}{\langle 24\rangle} \frac{1}{\langle 13\rangle\langle 34\rangle\langle 41\rangle} = \oint_{C_4} \langle 0|\phi_\lambda (\chi\phi)_1 \frac{[2P\lambda\rangle}{\langle 2\lambda\rangle} (\chi\phi)_3 (\chi\phi)_4 \chi_\lambda|0\rangle$$

$$= \oint_{C_4} \langle\lambda|(\chi\phi)_1 \frac{[2P\lambda\rangle}{\langle 2\lambda\rangle} (\chi\phi)_3 (\chi\phi)_4 |\lambda\rangle$$

Graviton amplitudes (cont'd.)

- The factor $\langle 12 \rangle^8$ can come from $\mathcal{N} = 8$ supersymmetry, the denominators $\langle 12 \rangle \langle 23 \rangle ...$ can come from Dirac propagators.
- This finally leads to (Nair)

$$\mathcal{A}(1, 2, \cdots, n) = \left[\frac{1}{2!} \frac{\delta}{\delta h_2} .. \frac{\delta}{\delta h_{n-2}} \frac{\delta}{\delta v_{n-1}} \frac{\delta}{\delta v_n} W[h, v, 1] \right]_{\bar{A}=0}$$

$$W[h, v, 1] = -\frac{4}{\kappa^2} \int d^4x d^{16}\theta \left[\oint_{C_n} \langle \lambda | V_1 \left(\frac{1}{\bar{\partial} - \bar{A}} \right)_{11} | \lambda \rangle \right]_{v_1=}$$

$$\bar{A} = V + \mathcal{E}.$$

Graviton amplitudes (cont'd.)

- V, $\mathcal{E}$ are the vertex operators

$$
\begin{aligned}
V &= -\frac{\kappa\pi}{2}\, v\, \chi\phi\ \exp(ip\cdot x + i\pi^A \theta^\alpha_A \eta^\alpha) \\
\mathcal{E} &= -\frac{\kappa\pi}{2}\, h\ \exp(ip\cdot x + i\pi^A \theta^\alpha_A \eta^\alpha)\ \frac{\bar{\pi}_{\dot{A}} \lambda^A (-i\nabla^{\dot{A}}_A)}{\langle \pi\lambda \rangle}
\end{aligned}
$$

- The nonholomorphic terms are Chan-Paton factors,

$$
D^A_{\dot{A}} = (\sigma^\mu)^A_{\dot{A}} \left[e^a_\mu \partial_a - \omega^{ab}_\mu J^{ab} \right] \ \approx \nabla^A_{\dot{A}} - \left(h^a \partial_a + \omega^{ab} J^{ab} \right)^A_{\dot{A}}
$$

$$
\sim \mathcal{E} \qquad \sim V
$$

J^{ab} contains a term like $\chi\phi$.

- Is there a 'twistor string' for $\mathcal{N} = 8$ supergravity?

New twistor strings

- (Abou-Zeid, Hull, Mason) It is possible to eliminate conformal gravitons using additional gauge freedom
- Modify the action as

$$S = \int [Y \bar{D} Z \; + \; \tilde{B}_i K_i] \; + \; S_C$$

where $K_i = k_{i\alpha} \partial Z^\alpha$.

- This has gauge invariance under

$$\delta \tilde{B}_i = \bar{\partial} \Lambda_i, \qquad \delta Z^\alpha = 0, \qquad \delta Y_\alpha = k_{i\alpha} \partial \Lambda_i + 2 \Lambda_i k_{i[\alpha,\beta]} \partial Z^\beta$$

- The anomalies are now

$$CD - N - 28 + C_C - 2(d - n), \qquad \kappa = D - N \sum_i \epsilon_i (h_i)^2$$

New twistor strings (cont'd.)

- There are many anomaly-free solutions, one of them seems like $\mathcal{N} = 8$ supergravity
- This corresponds to gauging with $\tilde{B}\, w(Z)\epsilon_{\dot{A}\dot{B}} U^{\dot{A}} \partial U^{\dot{B}}$, where $w(Z)$ has degree of homogeneity -2.
- The vertex operator $V_f = \int f^\alpha Y_\alpha$ is now changed; there is the restriction $\partial_\alpha f^\alpha = 0$ giving, eventually

$$V_f = \int d\sigma\ f^A Y_A = \int d\sigma\ \left[\epsilon^{AB} \frac{\partial h}{\partial W^A}\right]\ Y_A$$

- h is of degree of homogeneity 2,

$$h = h_2 + h^a_{\frac{3}{2}} \xi^a + \cdots + h_0 \xi^1 \xi^2 \xi^3 \xi^4$$

$\Longrightarrow \mathcal{N} = 4$ graviton multiplet with +helicity graviton.

New twistor strings (cont'd.)

- Similarly, the vertex operator $V_g = \int g_\alpha \partial Z^\alpha$ has further gauge symmetries

$$g_\alpha Z^\alpha = 0, \qquad g_\alpha \to g_\alpha + \partial_\alpha \chi, \qquad g_\alpha \to g_\alpha + \epsilon^{\dot{A}\dot{B}} U_{\dot{B}}\, \eta$$

- The solution is of the form

$$V_g = \int d\sigma \left(\frac{U \cdot A}{\bar{\Pi} \cdot A\, \beta \cdot \Pi} \right) \left[\beta_A \partial W^A - \beta \cdot W \frac{\bar{\Pi} \cdot \partial U}{\bar{\Pi} \cdot U} \right] \tilde{h}$$

$Z = (W, U, \xi)$ β is an arbitrary spinor.

- $\tilde{h}$ has the expansion $\tilde{h} = h_{-6} \xi^1 \xi^2 \xi^3 \xi^4 + \cdots$, giving $\mathcal{N} = 4$ graviton multiplet with the negative helicity graviton.
- There are some more vertex operators which can be used to complete the build-up of $\mathcal{N} = 8$ gravity multiplet.

Graviton amplitudes, again

- The vertex operators have no spacetime scaling dimensions, $[V_f] = [V_g] = 0$. This suggests we may not get Einstein supergravity.
- AHM calculated

$$\langle V_f(1) V_f(2) V_g(3) \rangle_{d=0} = \left(\frac{\langle 12 \rangle}{\langle 31 \rangle \langle 23 \rangle} \right)^2 \delta^{(4)}(p_1 + p_2 + p_3)$$

  This agrees with Einstein supergravity.
- For the $(+--)$ amplitude, I find

$$\langle V_f(1) V_g(2) V_g(3) \rangle_{d=0} = \langle V_f(1) V_g(2) V_g(3) \rangle_{d=1} = 0$$

- This suggest that it is some sort of chiral (antiselfdual) $\mathcal{N} = 8$ supergravity, like the one found by Siegel.

Further citations

Besides citations given, important work has been done by

- Bern, Kosower, Dixon, Bena, Del Luca,... (UCLA-SLAC-Saclay)
- Brandhuber, Spence, Travaglini, Bedford (Queen Mary)
- Khoze, Glover, Georgiou, ... (Durham)
- W. Siegel, ...
- Risager, Mansfield, ...
- many others