# Aspects of Integrability in $\mathcal{N}$ =4 SYM

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- Motivations and remarks about the AdS/CFT Conjecture

- Super-Yang Mills in Radial Quantization
  3+1 Gauge Theory $\rightarrow$0+1 Matrix QM

- Various Spin Systems as Different Limits of Matrix QM
  Intebrable Structures in SYM and Bethe Ansatzae

- Some Problems and Open Questions

- Conclusions

AdS/CFT Correspondence:

$\mathcal{N}$=4 SYM on $R \times S^3 \Longleftrightarrow$ IIB Strings on $AdS_5 \times S^5$

Spectrum of Anomalous Dimensions $\Longleftrightarrow$ String Spectrum

How Does the Gauge Theory Reorganize Itself as a String Theory?

Large $N$ Dilatation Operator of the Gauge Theory $\Longleftrightarrow$ Integrable Quantum Spin Chain in $1+1$ Dimensions.

Gauge Theory Spectrum from Spin Chains.

Spin Chain Behaves Like a String Sigma Model!

$\mathcal{N} = 4 SYM$ on $R \times S^3$

$S = \int d^4x tr \left[\mathcal{F}^2 + (D\Phi)^2 + g^2[\Phi^i, \Phi^j]^2 + Fermions\right]$

$$\mathcal{F}_{\mu\nu} = D_{[\mu}A_{\nu]}$$

$\beta_g = 0$ to all orders in perturbation theory.

Computation of anomalous dimensions $\rightarrow$ Radial Hamiltonian $\rightarrow$ Dilatation Operator $\rightarrow$ Matrix Quantum Mechanics $\rightarrow$ Various Integrable Dynamical Systems

$$D = D_0 + g^2 D_1 + g^4 D_2 + \cdots$$

$$Tr(\Phi^{i_1} \cdots \Phi^{i_m}) \longleftrightarrow Tr(a^{\dagger i_1} \cdots a^{\dagger i_m})|0>$$

$$D = Tr\left(a^{\dagger i} a_i + g^2 [a^{\dagger i}, a^{\dagger j}][a_i, a_j] + O(g^4)\right)$$

$$D = L + g^2 Tr[a^{\dagger i}, a^{\dagger j}][a_i, a_j] + O(g^4)$$

BPS Operators

$$Tr(Z^k), Tr(Z^m W) \longleftrightarrow Tr(a^{\dagger 1})|0>, Tr((a^{\dagger 1})^m a^{\dagger 2})|0>$$

Generic non-BPS Operators

$$Tr(a^{\dagger 1} a^{\dagger 1} a^{\dagger 2} a^{\dagger 2} a^{\dagger 1})|0>$$

$$(\Uparrow\Uparrow\Downarrow\Downarrow\Uparrow)$$

$$D = L + \lambda \sum_{i=1}^{L} (1 - P_{i,i+1}) + O(\lambda^2)$$

Heisenberg spin chain as the one loop (large $N$) dilatation operator. Concrete evidence of integrability beyond the protected sectors and very strong hints about the emergence of strings from the gauge theory.

Integrability of the Dilatation Operator:

$$D_1 = \lambda \sum_i (1 - S^a_b(i) S^b_a(i+1))$$

Non-Local Conserved Charges and Yangian Symmetry of $D_1$(Dolan, Nappi, Witten '03):

$$(\mathcal{Q}^0)^a_b = \sum_i S^a_b(i) : \qquad su(2) \text{ Generator}$$

$$(\mathcal{Q}^1)^a_b = \frac{1}{2} \sum_{i<j} \theta(i,j) \left( S^a_k(i) S^k_b(j) \right)$$

$$[(\mathcal{Q}^0)^a_b, (\mathcal{Q}^0)^c_d] = \delta^c_b (\mathcal{Q}^0)^a_d - \delta^a_d (\mathcal{Q}^0)^c_b$$

$$[(\mathcal{Q}^0)^a_b, (\mathcal{Q}^1)^c_d] = \delta^c_b (\mathcal{Q}^1)^a_d - \delta^a_d (\mathcal{Q}^1)^c_b$$

$$\Delta(\mathcal{Q}^0)^a_b = (\mathcal{Q}^0)^a_b \otimes \mathcal{I} + \mathcal{I} \otimes (\mathcal{Q}^0)^a_b$$

$$\Delta(\mathcal{Q}^1)^a_b = (\mathcal{Q}^1)^a_b \otimes \mathcal{I} + \mathcal{I} \otimes (\mathcal{Q}^0)^a_b$$
$$+((\mathcal{Q}^0)^a_d \otimes (\mathcal{Q}^0)^d_b - (\mathcal{Q}^0)^d_b \otimes (\mathcal{Q}^0)^a_d)$$

$\Delta$ = Algebra Homomorphism $\rightarrow$ Serre Relation

$$\theta(j,k)\theta(j,n) + \theta(j,k)\theta(n,k) - \theta(j,n)\theta(n,k) = \theta(j,k)$$

Transfer matrix

$$T(u) = e^{\wp \frac{1}{u} \sum_i S(i)} = \sum_i \frac{1}{u^n} T_n$$

$$T(u)^a_b = \mathcal{I} + \frac{1}{u} (\mathcal{Q}^0)^a_b + \frac{1}{u^2} (\mathcal{Q}^1)^a_b + \cdots$$

$$[T^{ab}_s, T^{cd}_{p+1}] - [T^{ab}_{p+1}, T^{cd}_s] = \left( T^{cb}_p T^{ad}_s - T^{cb}_s T^{ad}_p \right)$$

$$\Delta T^{ab} = \sum_d T^{ad} T^{db}$$

$$[D_1, T^{ab}_n] = 0$$

RTT Relations:

$$R(u-v)(T(u) \otimes \mathcal{I})(\mathcal{I} \otimes T(v)) = (\mathcal{I} \otimes T(v))(T(u) \otimes \mathcal{I}) R(u-v)$$

$$R = u\mathcal{I} - P$$

$$[tr(T_m), tr(T_n)] = 0$$

Complete Integrability of $D_1$

From Spins to Strings

Near BPS Long Operators: $\Delta \approx J$

$$\Psi = |\Uparrow\Uparrow\Uparrow\Uparrow \cdots \Downarrow \cdots \Uparrow \cdots \Downarrow \cdots \Uparrow>$$

Sigma Model for Ferromagnetic Spin Waves:

$$D_1 = \lambda \sum_l \vec{S}(l)\vec{S}(l+1) \rightarrow \frac{\lambda}{J^2}\int dx \vec{\eta}(x)\partial_x^2\vec{\eta}(x)$$

Strings on $R \times S^3 \in AdS_5 \times S^5$ with Large Angular Momentum $J$.

$$H = \frac{\lambda}{J^2}\int dx \vec{\eta}(x)\partial_x^2\vec{\eta}(x) + O((\frac{\lambda}{J^2})^2)$$

$$t_{String}(u) = e^{\frac{1}{u}\int dx \vec{\eta}(x).\vec{\sigma}} = \sum_n \frac{1}{u^n} t_n n$$

$$\{t_s^{ab}, t_{p+1}^{cd}\} - \{t_{p+1}^{ab}, t_s^{cd}\} = \left(t_p^{cb} t_s^{ad} - t_s^{cb} t_p^{ad}\right)$$

$$\{H, t_n\} = 0$$

The complete String Sigma Model Has a $psu(2,2|4)$ Yangian Symmetry (Bena, Polchinksi, Roiban '03)

Higher Loops in the $su(2)$ Sector

$$D = \sum_i ((1-P_{i.i+1}) + \lambda((1-P_{i.i+2}) - 4(1-P_{i.i+1})) + \ldots)$$

$$(\mathcal{Q}^0))^a_b = \sum_i S^a_b(i)$$

$$(\mathcal{Q}^1))^a_b = (\mathcal{Q}^1_1))^a_b + \lambda(\mathcal{Q}^1_2))^a_b + \cdots$$

$$\theta(i,j) = \frac{t^{-i}}{t^{-i} - t^{-j}}$$

$$t = \sum_{n=1}^{\infty} c_n \lambda^n$$

$$c_1 = 1, c_2 = -3, c_3 = 14$$

Establishes 3 loop Yangian invariance.
3 Loops in the $su(2)$ sector (Agarwal and Rajeev '04)
2 loops in the $su(1|1)$ sector (Agarwal, '05)
2 loops in the $su(2|1)$ sector (Zwiebel, '06)

The One Loop Bethe Ansatz

2 magnon state:

$$\sum_{x<y} \Psi(x,y)| \Uparrow\Uparrow\Uparrow\Uparrow\Downarrow\Uparrow\Uparrow\Uparrow\Uparrow\Downarrow\Uparrow \cdots \Uparrow>$$

$$\Psi(x,y) = e^{i(p_1 x + p_2 y)} + S(p_1,p_2) e^{i(p_1 y + p_2 x)}$$

2 body $S$ matrix

$$S(p_1,p_2) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}$$

2body Bethe equations

$$e^{ip_1 L} = S(p_1,p_2), e^{ip_2 L} = S(p_2,p_1)$$

Many body Bethe equations

$$e^{ip_k L} = \prod_{j \neq k} S(p_k,p_j)$$

Equations in terms of Bethe roots:

$$\phi(p) = \frac{1}{2}\cot(\frac{p}{2}), e^{ip_k L} = \prod_{j \neq k} \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i}$$

$$E(p) = 4\sin^2(\frac{p}{2})$$

From one loop to all loops in the $su(2)$ sector:

$$D = \sum_i ((1 - P_{i.i+1}) + \lambda((1 - P_{i.i+2}) - 4(1 - P_{i.i+1})) + \ldots)$$

One loop Bethe ansatz:

$$e^{ip_k L} = \prod_{j \neq k} \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i} e^{iQ(p_k, p_j)}$$

Bethe roots/phase shifts:

$$\phi(p) = \frac{1}{2} \cot(\frac{p}{2})$$

Dispersion relation:

$$E(p) = 4 \sin^2(\frac{p}{2})$$

All loops Bethe roots:

$$\phi(p) = \cot(\frac{p}{2}) \sqrt{1 + \lambda \sin^2(\frac{p}{2})}$$

All loops Dispersion relations

$$E(p) = \frac{1}{\lambda}(\sqrt{1 + 4\lambda \sin^2(\frac{p}{2})} - 1)$$

Beisert, Dippel, Serban and Staudacher '04

Is there a real underlying spin chain that produces the same Bethe equations and dispersion relations:

Inozemtsev/Hyperbolic spin chain (Serban and Staudacher '04)

Hubbard model at strong coupling (Rej, Serban and Staudacher '05)

Results have been generalized from the small $su(2)$ sector to the full gauge theory (Beisert and Staudacher '05).

BPS/Protected Operators, e.g

$$Z = \Phi^1 + i\Phi^2, TrZ^k, TrZ^m\Psi, Det[Z]$$

$$D = D_0 + g^2 D_1 + \cdots$$

$$D_{BPS} = D_0$$

(Gauged) Matrix Harmonic Oscillators

$$D = Tr(A^\dagger A + B^\dagger B), A, B \leftrightarrow Z, \Psi$$

BPS Sector of the gauge theory $\rightarrow$ Supergravity/Weak coupling limit of string theory.

$$D = Tr(A^\dagger A + B^\dagger B), A, B \leftrightarrow Z, \Psi^1$$

$$A = X + iP, X = U^\dagger x U$$

$$(b)^i_j = (UBU^\dagger)^i_j$$

$$D = \sum_i \left( -\frac{\partial}{\partial x_i^2} + x_i^2 \right) + \sum_{i \neq j} \left( \frac{\mathcal{L}^i_j \mathcal{L}^j_i}{(x_i - x_j)^2} \right) + trb^\dagger b$$

$$\mathcal{L}^i_j = U^i_m \frac{\partial}{\partial U^m_j}, [\mathcal{L}^i_j, \mathcal{L}^k_l] = \delta^k_j \mathcal{L}^i_l - \delta^i_l \mathcal{L}^k_j$$

$$\Psi^{i_1 \cdots i_n}_{j_i \cdots j_n}(x) \Pi^n_{k=1} (b^\dagger)^{j_k}_{i_k} |0>$$

$$\mathcal{L}^i_j = \sum_\beta \left( (b^\dagger)^i_l (b)^l_j - (b^\dagger)^l_j (b)^i_l \right)$$

Euler Calogero Model

Complete (Gauge Fixed) Description of all $su(2|3)$ BPS Dynamics

$$\prod_i tr(A^{\dagger n_i})|0>$$

$$\mathcal{L}^i_j = 0$$

Revert Back to the Free Fermion Picture

Multi-Charge BPS operators: SUSY Descendants

$$Q = tr(b^{\dagger\alpha} a)$$

$$(a)^i_j = \left(x_i + \frac{\partial}{\partial x_i}\right)\delta^i_j + \frac{(1-\delta^i_j)\mathcal{L}^i_j}{x_i - x_j}$$

Non-BPS Excitations → Gauge Fixed Description of Operators of the Gauge Theory at $g^2_{YM} = 0$.

Restriction to BPS States

$$\frac{1}{\sqrt{N^{n_1+1}}}tr\left((A^\dagger)^{n_1}B^\dagger\right)\frac{1}{\sqrt{N^{n_2}}}tr\left((A^\dagger)^{n_2}\right)\cdots$$

$$\cdots\frac{1}{\sqrt{N^{n_i+1}}}tr\left((A^\dagger)^{n_i}B^\dagger\right)|0>$$

Protected in the Large $N$ limit

BMN Like Near Chiral Primaries at finite $N$

$$\prod_m \Psi(x_1\cdots x_N)^{i_1\cdots i_m}(b^\dagger)_{i_1}\cdots(b^\dagger)_{i_m}|0> +O(\frac{1}{N})$$

$$(b^{\dagger\alpha})_i = (b^{\dagger\alpha})^i_i.$$

$\mathcal{L}^i_j\mathcal{L}^j_i = \frac{1}{2}(1-\Pi_{i,j})$

$$D=\sum_i\left(-\frac{1}{2}\frac{\partial}{\partial x_i^2}+b^{\dagger i}b_i+\frac{1}{2}x_i^2\right)+\frac{1}{2}\sum_{i\neq j}\left(\frac{1-\Pi_{i,j}}{(x_i-x_j)^2}\right).$$

Rational Super-Calogero Model

AA and A.P.Polychronakos ('06)

The Map: Introduce Bosonic and Fermionic Oscillators $B_n$, $F_k$ with frequencies given by $n$ and $k$

$$(B_n)^k|0> \leftrightarrow \left[tr(A^\dagger)^n\right]^k|0>$$

$$(F_n)^k|0> \leftrightarrow \left[tr(A^\dagger)^{n-1}b^\dagger\right]^k|0>$$

$[F_m, F_n]_+ = 0, [B_m, F_n] = 0, [B_m, B_n] = 0$
$[Q, F_m]_+ = 0, [Q^\dagger, F_n]_+ = B_n, [H, F_n] = nF_n$
$[Q, B_n] = 2nF_n, [Q^\dagger, B_n] = 0, [H, B_n] = nB_n$
Spectrum Generating Algebra of the Rational Super-Calogero Model

(Freedman and Mende '90)

$$e^W(H_{Calogero})e^W = H_{Super-Oscillators}$$

Degeneracies etc

$$B_n|0> \text{ and } \prod_{i=l}^{l} B_{n_i}|0>, \sum_i n_i = n$$

$$tr[(A^\dagger)^n]|0> \text{ and } \prod_i \left[tr(A^\dagger)^{n_i}\right]|0>, \sum_i n_i = n.$$

State of Energy m $\to$ Young Diagram with Columns of Length $n_1 \geq n_2 \geq \cdots \geq n_i$, $n_1 + \cdots + n_i = m$

Further Degeneracies

$$B_m|0> \leftrightarrow (Q_m)|0>= F_m|0>$$

Hidden Symmetries and Non-Local Conservation Laws

The Lax Operator:

$$L_{j,k} = \delta_{j,k}\frac{\partial}{\partial x_j} + \hbar(1-\delta_{j,k})\theta(j,k)\Pi_{j,k}$$

and

$$\theta(j,k) = \frac{e^{-\frac{\hbar}{2}(x_i - x_j)}}{\sinh\frac{\hbar}{2}(x_i - x_j)}$$

Non-Local Conserved Charges:

$$T_n^{ab} = \sum_{j,k} S^{ab}(j)(L^n)_{j,k}$$

$T_n^{ii}$ are conserved.

Hamiltonian at the Center of the Algebra

$$\frac{1}{2}\sum_{j,k}(-\partial_j^2 + x_j^2 + b^\dagger(j)b(j) + \hbar\Pi_{j,k}\partial_j\theta(j,k) + \hbar^2\theta_{j,k}\theta_{k,j})$$

Yangian Algebra of the Conserved Currents

$$[T_s^{ab}, T_{p+1}^{cd}]_\pm - [T_{p+1}^{ab}, T_s^{cd}]_\pm =$$
$$\hbar(-1)^{\epsilon(c)\epsilon(a)+\epsilon(c)\epsilon(b)+\epsilon(b)\epsilon(a)} \left(T_p^{cb}T_s^{ad} - T_s^{cb}T_p^{ad}\right)$$

Rational Calogero Limit $\hbar \rightarrow 0$

Yangian Algebra $\rightarrow$ Loop Algebra

$$[T_s^{ab}, T_p^{cd}]_\pm = \delta_{b,c}T_{p+s}^{ad} - (-1)^{(\epsilon(a)+\epsilon(b))(\epsilon(c)+\epsilon(d)}\delta_{a,d}T_{p+s}^{cb}$$

SUGRA Limit of String Theory $\rightarrow$ Classical Limit of the Hidden Symmetries

$$T_1^{21} = Q, T_1^{12} = Q^\dagger$$

$$H = [T_1^{21}, T_1^{12}]_+$$

$$H_{n+m} = [T_n^{12}, T_m^{21}]_+$$

$$[T_m^{11}, T_n^{11}] = [T_m^{11}, T_n^{22}] = [T_m^{22}, T_n^{22}] = 0 \forall m, n$$

Baryonic operators and open spin chains:

Single trace operators

$$Tr(Z^n), Tr(ZWWZZZWZWWW)etc$$

are described by closed/periodic spin chains. The gauge theory also has 'Baryonic' operators of size of $O(N)$, such as

$$\epsilon_{i_1\cdots i_N}\epsilon^{j_1\cdots j_N}(Z^{i_1}_{j_1}\cdots Z^{i_N}_{J_N})$$

and non-BPS excitations such as

$$\epsilon^{j_1\cdots j_{N-1}j_N}_{i_1\cdots i_{N-1}i_N}((Z^{i_1}_{j_1}\cdots Z^{i_{N-1}}_{J_{N-1}})(WZZWZWW\cdots W)^{i_N}_{j_N}$$

Such open chains are closed under dilatation in the large $N$ limit.

$$D_1 = \sum_{l=1}^{L-1}(\lambda)Q^Z_1 Q^Z_J(I - P_{l,l+1})Q^Z_1 Q^Z_J$$

The One Loop Bethe Ansatz:

Ground State:

$$\epsilon^{j_i \cdots j_{N-1} j_N}_{i_i \cdots i_{N-1} i_N} (Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{j_{N-1}})(WWWW \cdots WWW)^{i_N}_{j_N}$$

2 Magnon State

$$|\Psi_2 >= \sum_{x<y} \Psi(x,y)|x,y>$$

with

$$\Psi(x,y) = \sum_p \sigma(p) A(k_i,k_2) e^{i(k_1 x_1 + k_2 x_2)}$$

$$\frac{\alpha(k_i)\beta(k_i)}{\alpha(-k_i)\beta(-k_i)} = \prod_{j \neq i} \frac{S(-k_i,k_j)}{S(k_i,k_j)}$$

$$\begin{aligned} \alpha(-k) &= e^{2ik} - e^{ik} \\ \beta(k) &= e^{i(L-1)k} - e^{iLk} \end{aligned}$$

$$S(k_1,k_2) = 1 - 2e^{ik_2} + e^{i(k_1+k_2)}$$

$$E = 4\lambda \sum_i \left( \sin^2(\frac{k_i}{2}) \right)$$

Surprises at two loops:

$$D_2 = \sum_l Q_1^Z Q_J^Z ((I - P_{l,l+1}) +$$
$$\lambda((I - P_{l,l+2}) - 4(I - P_{l,l+1}))Q_1^Z Q_J^Z$$

No longer an integrable perturbation to the one loop Hamiltonian. Three body processes e.g in

$$\epsilon^{j_1 \cdots j_{N-1} j_N}_{i_1 \cdots i_{N-1} i_N}((Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{J_{N-1}})(WZZWW \cdots W)^{i_N}_{j_N}$$

do not factorize into two body ones.(AA '06)

Higher loop integrability in the Baryonic sector of the gauge theory is an open question!