Worldsheet Scattering and AdS/CFT

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Motivations

- There has been a good deal of success recently in comparing the energies of semi-classical string solutions with the anomalous dimensions of gauge invariant operators in the context of AdS/CFT (in the planar limit).
- Much of the progress has involved considering specific string solutions e.g. the solutions with certain large charges of Frolov, Tseytlin, GKP, BMN and many others.
- In particular the (possible) presence of integrability in the classical (quantum) worldsheet theory (Bena, Polchinski, Roiban also KMMZ and others) and in the dual gauge theory (Minahan&Zarembo) has led to the introduction of a number of powerful tools e.g. the Bethe ansatz.
- The S-matrix seems to be a particularly simple tool to describe the system and there has been significant progress in finding the correct S-matrix for the gauge theory and worldsheet theory using symmetries, generalised crossing, perturbative results, wild conjectures,…(AFS, BDS, HL, FK,RTT, Janik, Beisert, Staudacher, HM, **BHL/BES**,…)
- Of course it would be useful to have a direct way to calculate and test the various conjectured S-matrices and in this talk I will try to outline a few partial results regarding the worldsheet S-matrix in the small momentum limit.
- Based on hep-th/0611169 with T. Klose, R. Roiban, and K. Zarembo and work in progress (with T. Klose, J. Minahan and K. Zarembo) .

Outline

- Briefly outline Metsaev & Tseytlin Green-Schwarz string theory on supercoset space

$$\frac{\mathrm{PSU}(2,2|4)}{SO(4,1)\times SO(5)}$$

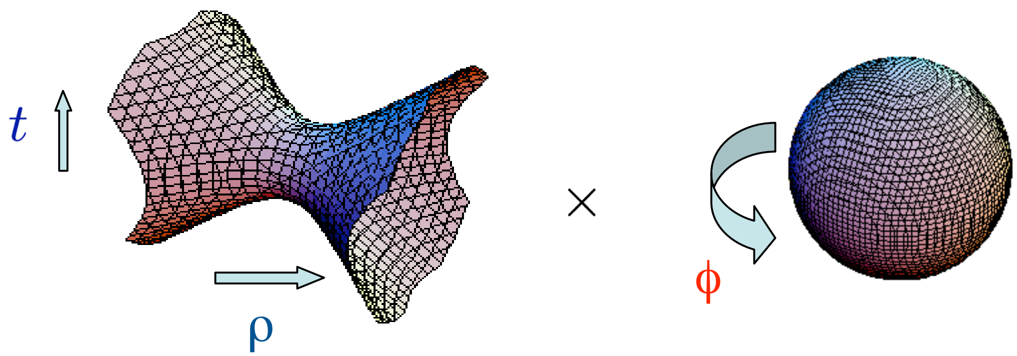
  and in particular the construction of a gauge fixed light-cone action.
- Describe the calculation of the classical string S-matrix in light-cone gauge. This calculation leads to several puzzles, in particular the resulting S-matrix appears be inconsistent with underlying symmetries: outline and explain how this issue is resolved, in particular the appearance of a non-trivial coproduct is important.
- Compare results with spin chain description of dual YM theory and the S-matrix for the dynamical SU(2|2) sector.
- Discuss the "near-flat space" model, the higher loop calculation (up to two-loops) for this model and how it compares with full string S-matrix. We also describe how the symmetries of the theory are realised in this limit and notably the restoration of "Lorentz invariance" and super-symmetry.
- Describe the one-loop calculation for the full theory in the near-plane wave limit and outline difficulties, in particular unforseen divergences
- Conclusions and possible future directions.

$AdS_5 \times S^5$ Geometry

The metric in global coordinates is given by:

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2)$$

where R denotes the radius of both the AdS_5 and S^5 subspaces, and $d\Omega_3^2$, $d\tilde{\Omega}_3^2$ are 3-spheres. The coordinate ϕ has periodicity 2π.



- We must use the covering space so that t is not periodic
- Bosonic isometry group, SO(4,2) × SO(6), combines with the supersymmetries into the supergroup PSU(2,2|4)

- The string action in this background is

$$S = \frac{R^2}{4\pi} \int d^2\sigma \ (\sqrt{-h} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N) + \mathrm{fermions}$$

- Action is non-linear and quantization of string theory in this background is extremely difficult & so far unsolved.
- It is only tractable in certain limits or when expanding about certain classical solutions e.g. spinning string solutions (Frolov, Tseytlin,…) or the fast point-like string moving on a geodesic on the sphere (BMN).
- Can describe the target space as a (super)-coset which allows for a simple (relatively speaking) description of the action including the fermions.
- Just like for sigma models on coset spaces this action is classically integrable (it has an infinite number of (non-local) worldsheet conserved currents) which gives us hope that it might be solvable.

String S-matrix

- First we discuss expanding the Metsaev & Tseytlin action for the GS string about the BMN vacuum and using the light-cone Lagrangian to find the S-matrix. The covariant action is

$$\mathcal{S} \;=\; -\frac{1}{2}\int_{\partial M_3} d^2\sigma\;\sqrt{g}g^{ab}L^{\mu}_{a}L^{\mu}_{b} + i\int_{M_3} s^{IJ}L^{\mu}\wedge \bar{L}^{I}\Gamma^{\mu}\wedge L^{J}$$

- The L^{μ} and L^{α} are the bosonic and fermionic components of the super-vielbien
- $AdS_5 \times S^5$ can be written as a coset manifold which makes finding the vielbien possible (Kallosh, Rahmfeld, Rajaraman)

$$L^{\alpha J}_{b} \;=\; \frac{\sinh \mathcal{M}}{\mathcal{M}}\mathcal{D}_b\theta^{\alpha J}$$

$$L^{\mu}_{a} \;=\; e^{\mu}{}_{\rho}\partial_a x^{\rho} - 4i\bar{\theta}^{I}\Gamma^{\mu}\left(\frac{\sinh^2(\mathcal{M}/2)}{\mathcal{M}^2}\right)\mathcal{D}_a\theta^{I}$$

with $$(\mathcal{D}_a\theta)^I \;=\; \left(\partial_a\theta + \frac{1}{4}\left(\omega^{\mu\nu}{}_{m}\,\partial_a x^m\right)\Gamma^{\mu\nu}\theta\right)^I - \frac{i}{2}\epsilon^{IJ}e^{\mu}{}_{m}\,\partial_a x^m\Gamma_*\Gamma^{\mu}\theta^J$$

$$(\mathcal{M}^2)^{IL} \;=\; \epsilon^{IJ}(\Gamma_*\Gamma^{\mu}\theta^J\bar{\theta}^L\Gamma^{\mu}) + \frac{1}{2}\epsilon^{KL}(-\Gamma^{jk}\theta^I\bar{\theta}^K\Gamma^{jk}\Gamma_* + \Gamma^{j'k'}\theta^I\bar{\theta}^K\Gamma^{j'k'}\Gamma'_*)$$

Worldsheet Action

- There are several issues involved in finding the appropriate action.

  General Outline:
- Introduce light-cone coordinates and make gauge choice.
- We fix $x^+ = \tau$, $p_- = 1$ and using kappa-symmetry $\Gamma^+\theta = 0$
- Remove x^- using the constraint equations from the metric equations of motion (actually leaves zero mode undetermined) .
- Determine the world sheet metric using the x^- equation of motion
- We calculate the light-cone Hamiltonian and express it in terms of the transverse coordinates & momenta
- $$-P_+ \equiv H_{l.c.}(p^I, x'^I, x^I, \rho, \psi', c.c)$$
- At this point it is necessary to make a redefinition of the fermions in order to get canonical Poisson brackets.
- Legendre transform in the transverse directions to find the light-cone Lagrangian.
- Expand in inverse powers of $\sqrt{\lambda}$.

Light-cone gauge and expansion parameters

This leads us to the issue of different gauge choices: a useful interpolating choice is (AFZ)

$$P_- = (1-a)E + aJ$$

- If we choose $J=P_-$ to be constant and this gauge choice should give a s-matrix which agrees with the small momentum limit of AFS (once we take into account the difference in the definition of the string length).
- The "uniform" light cone-gauge, corresponding to E+J constant, the formula are a little simpler and the scattering matrix should agree with that of Frolov, Plefka and Zamaklar. Of course in the end these different choices should give equivalent physical descriptions.
- Another issue is the exact expansion parameters: Gauge choice fixes string length

$$\mathcal{J} = \frac{2\pi}{\sqrt{\lambda}} P_-$$

one can now take $\mathcal{J}$ to infinity, which allows a sensible definition of the S-matrix and $\frac{2\pi}{\sqrt{\lambda}}$ as the loop counting parameter.

Equivalently one can rescale the world-sheet coordinate by $\sqrt{\lambda}$ the world-sheet length is P_- which we take to infinity and now take a small momentum expansion. At least at tree-level these two expansions should be equivalent.

PSU(2,2|4) String S-matrix

- After gauge fixing physical fields form a bi-fundamental representation of psu(2|2)$_L$ x psu(2|2)$_R$ (with shared center)

$$
\begin{array}{ccc}
Y_{a\dot{a}} & \leftrightarrow & \Psi_{a\dot{\alpha}} \\
\updownarrow & & \updownarrow \\
\Upsilon_{\alpha\dot{a}} & \leftrightarrow & Z_{\alpha\dot{\alpha}}
\end{array}
$$

- Two particle S-matrix acts on the tensor product of the su(2|2)^2 module W_p

$$\mathbb{S} : W_p \otimes W_{p'} \to W_p \otimes W_{p'}$$

- The expectation is that S-matrix describes an integrable system which implies that
  - That it splits into two factors, one for each psu(2|2) $\mathbb{S} = \mathbf{S} \otimes \mathbf{S}$
  - That there is no particle production
  - The the multiparticle S-matrix factorizes into two-particle S-matrices which in turn satisfy the YBE.
  - To lowest order

$$\mathbb{S}(p,p') = \mathbb{1} + \frac{2\pi i}{\sqrt{\lambda}} \frac{\mathbb{T}(p,p')}{\varepsilon' p - \varepsilon p'} + O\left(\frac{1}{\lambda}\right)$$

- T-matrix also factorises as $\mathbb{T} = \mathbb{1} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbb{1}$
  and satisfies the cYBE.
  Using the manifest su(2)^4 symmetries we can write the T-matrix as

$$
\begin{aligned}
\mathbf{T}_{ab}^{cd} &= A\,\delta_a^c\delta_b^d + B\,\delta_a^d\delta_b^c\ , & \mathbf{T}_{ab}^{\gamma\delta} &= C\,\epsilon_{ab}\epsilon^{\gamma\delta}\ ,\\
\mathbf{T}_{\alpha\beta}^{\gamma\delta} &= D\,\delta_\alpha^\gamma\delta_\beta^\delta + E\,\delta_\alpha^\delta\delta_\beta^\gamma\ , & \mathbf{T}_{\alpha\beta}^{cd} &= F\,\epsilon_{\alpha\beta}\epsilon^{cd}\ ,\\
\mathbf{T}_{a\beta}^{c\delta} &= G\,\delta_a^c\delta_\beta^\delta\ , & \mathbf{T}_{\alpha b}^{\gamma d} &= L\,\delta_\alpha^\gamma\delta_b^d\ ,\\
\mathbf{T}_{a\beta}^{\gamma d} &= H\,\delta_a^d\delta_\beta^\gamma\ , & \mathbf{T}_{\alpha b}^{c\delta} &= K\,\delta_\alpha^\delta\delta_b^c\ .
\end{aligned}
$$

- This leads to a puzzle:
  - T-matrix does not commute with global symmetries
  - Resolution: Existence of additional central charges and the global charges are non-local in light-cone gauge i.e. they have a non-trivial coproduct.

- First the extra central charges (Beisert & AFPZ): The algebra when realized in terms of Noether charges only closes up to constraint equations (or if all gauge invariances are fixed) up to compensating transformations. In the case at hand the level matching constraint is imposed on states and the Poisson bracket of susy charges closes up this constraint.

$$Tr\{Q,Q\} \propto \int d\sigma\,(P^I \acute{X}^I + i\Upsilon^* \acute{\Upsilon}) = -\int d\sigma \acute{X}^-$$

- Secondly there is non-trivial braiding for global charges: Quite generally currents in 1+1 dim field theory can have non-trivial braiding relations with fields

$$J^A{}_B(x)\,\Phi^C(y) = \Theta^{ACF}_{BDE}\,\Phi^D(y)J^E{}_F(x)$$

where there is an implicit time ordering of fields. (Fields at a later time lie to the left).

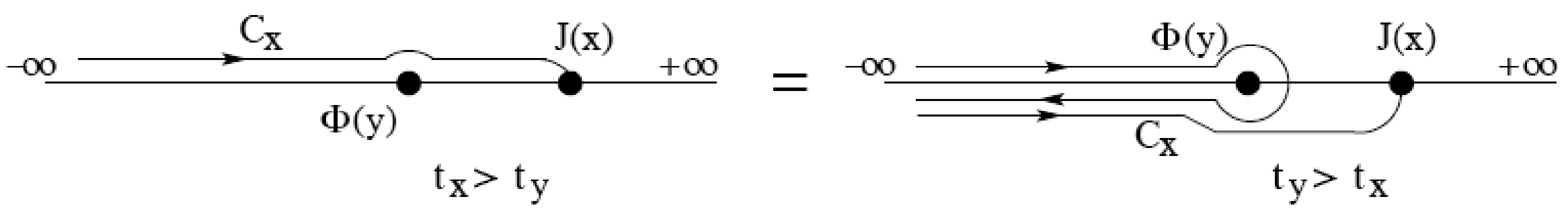
$$J(x)\,\Phi(y) \equiv J(x,t+\epsilon)\,\Phi(y,t)\Big|_{\epsilon\longrightarrow 0}$$

- When fields are mutually non-local and their definition requires a choice of contour one must be careful to deform the contour when reordering fields.

- For example if we consider the bilocal current in PCM's

$$J_{(2)} = *J_{(1)} + \frac{1}{2}[J_{(1)}, \chi] \qquad \chi = \int_\gamma *J_{(1)}$$

- The non-trivial braiding can be schematically shown by the contour argument



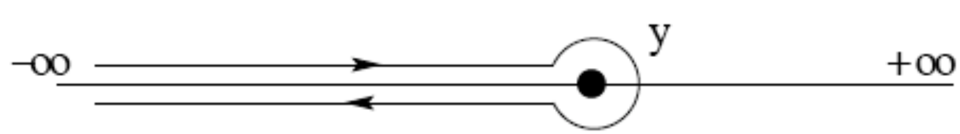
which results in the expression

$$J^a_{(2)}(x)\,\Phi(y) = \Phi(y)\,J^a_{(2)}(x) - \tfrac{1}{2}f^{abc}\widehat{Q}^b_{(1)}(\Phi(y))\,J^c_{(1)}(x)$$

where the action of the charge on a field is given by

$$\widehat{Q}^b_{(1)}(\Phi(y)) = \int_{\gamma_y} dz J^b_{(1)}(z)\Phi(y)$$

and where the contour is



which reduces to the usual commutator when the current is local.

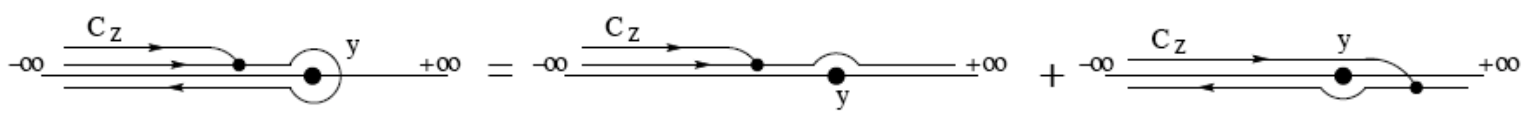
- For the worldsheet theory the global susy Noether currents depend explicitly on x⁻ rather than it’s derivatives: $J_Q = e^{\frac{i}{2}x^-}\tilde{J}_Q(X,\Upsilon)$

- One can then use the constraints to show that

$$x^-(x) = \int_{C_x} dw\, \acute{x}^-(w) \qquad \{\acute{x}^-(w),\, \Phi(y)\} = i\,\frac{2\pi}{\sqrt{\lambda}}\delta(w-y)\,\acute{\Phi}(y)$$

and so $$J_{Q^A{}_B}(x)\Phi(y) = \left(e^{-1/2\sigma_{AB}\partial_y}\Phi(y)\right) J_{Q^A{}_B}(x) \qquad x > y$$

- Can now calculate the action of the charges on fields by integrating along C_y using the contour manipulation



- Implying the result

$$\widehat{Q}^A_{(1)\,B}(\Phi^C(x)) = Q^A_{(1)\,B}\Phi^C(x) - \left(e^{-1/2\sigma\partial_x}\Phi^C(x)\right) Q^A_{1\,B}$$

- The other charges don’t involve any countour in their definition however as always to define the higher (non-local) conserved charges we will introduce extra contours as before.

- Similarly we can calculate the action on tenor products and it's easy to see that the charges do not satisfy the usual Leibnitz rule but rather have a non-trivial coproduct

$$\Delta(\hat{Q}_{(1)}) = \hat{Q}_{(1)} \otimes \mathbb{I} + e^{\frac{ip}{2}} \mathbb{I} \otimes \hat{Q}_{(1)}$$

- The other "local" charges have a trivial coproduct though as mentioned the higher conserved charges will be more complicated (in principle gives a definition for all powers and combinations of operators).
- This coproduct is similar to that constructed by considerations of the gauge theory Gomez & Hernandez, PST:
  - In fact using a nonlocal field redefinition they can be made the same where Z's are the length changing operators: $\phi^a \to \phi^a \mathcal{Z}^{\frac{1}{2}}$
  - On both sides the non-trivial braiding is due to the length changing operators (loosely defined): $e^{\pm ix^-} \leftrightarrow \mathcal{Z}^{\pm}$
- With a canonical choices for the counit (and unit and mult.)

$$\epsilon(x) = 0 \ \forall \ x \in \mathfrak{g} \qquad \epsilon(1) = 1$$

there is a unique choice for the antipode γ s.t.

$$\mathcal{M} \circ (\mathrm{id} \times \gamma) \circ \Delta = \eta \circ \epsilon = \mathcal{M} \circ (\gamma \times \mathrm{id}) \circ \Delta$$

We now go ahead and calculate the S-matrix to lowest order in $\sqrt{\lambda}$ (actually rescaled coefficients of T-matrix)

$$\mathrm{T}|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle = \mathrm{A}(p,p')|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle + \mathrm{B}(p,p')|Y_{b\mathrm{i}}Y'_{a\mathrm{i}}\rangle + \mathrm{C}(p,p')\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle$$

$$\mathrm{T}|Y_{a\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle = \mathrm{G}(p,p')|Y_{a\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle + \mathrm{H}(p,p')|\Upsilon_{\beta\mathrm{i}}Y'_{a\mathrm{i}}\rangle$$

$$\mathrm{T}|\Upsilon_{\alpha\mathrm{i}}Y'_{b\mathrm{i}}\rangle = \mathrm{K}(p,p')|Y_{b\mathrm{i}}\Upsilon'_{\alpha\mathrm{i}}\rangle + \mathrm{L}(p,p')|\Upsilon_{\alpha\mathrm{i}}Y'_{b\mathrm{i}}\rangle$$

$$\mathrm{T}|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle = \mathrm{D}(p,p')|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle + \mathrm{E}(p,p')|\Upsilon_{\beta\mathrm{i}}\Upsilon'_{\alpha\mathrm{i}}\rangle + \mathrm{F}(p,p')\epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle$$

$$A(p,p') = \frac{1}{4}\Big[(1-2a)\,(\varepsilon' p - \varepsilon p')^2 + (p-p')^2\Big]$$

$$B(p,p') = -E(p,p') = pp'$$

$$C(p,p') = F(p,p') = \frac{1}{2}\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,(\varepsilon' p - \varepsilon p' + p' - p)$$

$$D(p,p') = \frac{1}{4}\Big[(1-2a)\,(\varepsilon' p - \varepsilon p')^2 - (p-p')^2\Big]$$

$$G(p,p') = L(p',p) = \frac{1}{4}\Big[(1-2a)\,(\varepsilon' p - \varepsilon p')^2 - p^2 + p'^2\Big]$$

$$H(p,p') = K(p,p') = \frac{1}{2}pp'\frac{(\varepsilon+1)\,(\varepsilon'+1) - pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}}\ .$$

- Notable features:
  - invariant w.r.t. global charges only when we use non-trivial coproduct and where now algebra includes extra central charges. In terms of oscillators we have for example at quadratic level:

$$\mathfrak{Q}_\alpha{}^b = \int \frac{dp}{2\pi}\,(-)^{[\dot C]}\left[u\, c^\dagger_{\alpha\dot C}\, c^{b\dot C} - v\, c^{\dagger b\dot C}\, c_{\alpha\dot C}\right] \qquad \mathfrak{P} = \int \frac{dp}{2\pi}\, p\, c^\dagger_{A\dot A}\, c^{A\dot A}$$

  - such that

$$\{\mathfrak{Q}_\alpha{}^a, \mathfrak{Q}_\beta{}^b\} = -\tfrac{1}{2}\epsilon_{\alpha\beta}\epsilon^{ab}\,\mathfrak{P}\ ,$$
$$\{\mathfrak{S}_a{}^\alpha, \mathfrak{S}_b{}^\beta\} = -\tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,\mathfrak{P}\ ,$$
$$\{\mathfrak{Q}_\alpha{}^a, \mathfrak{S}_b{}^\beta\} = \delta_\alpha^\beta\,\mathfrak{L}_b{}^a + \delta_b^a\,\mathfrak{R}_\alpha{}^\beta + \tfrac{1}{2}\delta_\alpha^\beta\delta_b^a\,\mathfrak{H}$$

and the S-matrix should satisfy the intertwiner condition

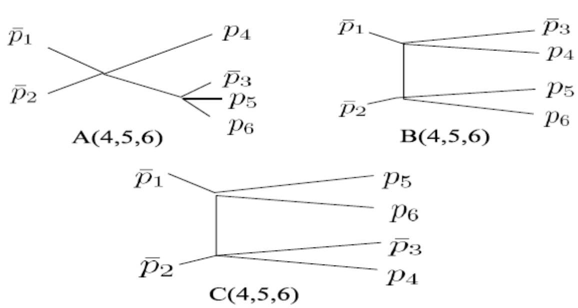
$$\left(\mathbb{1}\otimes \widehat{Q}_{(1)}{}^A{}_B + \widehat{Q}_{(1)}{}^A{}_B \otimes e^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}}p'}\,\mathbb{1}\right) S = S\left(\widehat{Q}^A_{(1)B}\otimes\mathbb{1} + e^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}}p}\,\mathbb{1}\otimes\widehat{Q}^A_{(1)B}\right)$$

which at tree-level corresponds to

$$[\mathfrak{Q}_\alpha{}^b\otimes\mathfrak{F} + \mathfrak{F}\otimes\mathfrak{Q}_\alpha{}^b, \mathrm{T}] = +(\tfrac{1}{2}\mathfrak{P}\mathfrak{F})\otimes\mathfrak{Q}_\alpha{}^b - \mathfrak{Q}_\alpha{}^b\otimes(\tfrac{1}{2}\mathfrak{P}\mathfrak{F})$$
$$[\mathfrak{S}_a{}^\beta\otimes\mathfrak{F} + \mathfrak{F}\otimes\mathfrak{S}_a{}^\beta, \mathrm{T}] = -(\tfrac{1}{2}\mathfrak{P}\mathfrak{F})\otimes\mathfrak{S}_a{}^\beta + \mathfrak{S}_a{}^\beta\otimes(\tfrac{1}{2}\mathfrak{P}\mathfrak{F})$$

and this is indeed satisfied.

- At least for the rank one sector involving a single boson we can explicitly show the absence of 2 ->4 particle production or equivalently the factorisation of three body scattering



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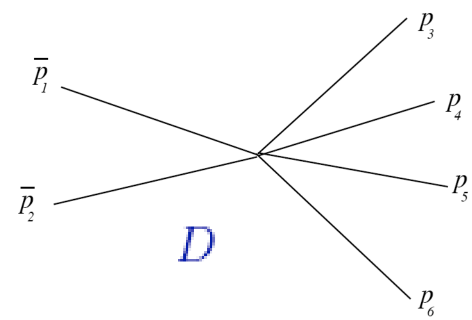


= 0

- The T-matrix does satisfy the cYBE

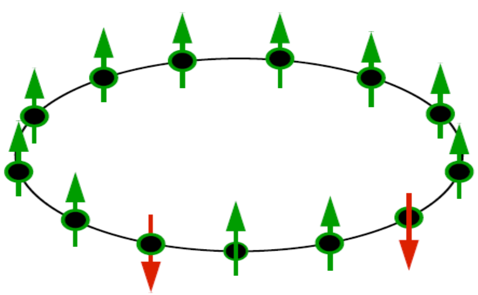
Spin Chains

- In calculating anomalous dimensions it is useful to shift focus from two-point functions to the dilatation generator acting on the space of local, gauge invariant operators

- Consider the space of single trace operators consisting of two complex scalars Z & W

$$\mathcal{O} = \mathrm{Tr}(ZZWZ\cdots ZWZ)$$

- Identify $Z = |\uparrow\rangle \quad W = |\downarrow\rangle \qquad \mathcal{O} = |\uparrow\uparrow\downarrow\uparrow\cdots\uparrow\downarrow\uparrow\rangle$



The $\mathcal{N}$=4 SYM Spin Chain

- At weak coupling planar U(N) gauge theory can be described by a spin chain. In particular the eigenvalue problem for the dilatation operator is related to that of a spin chain Hamiltonian

$$\boxed{D \cdot \mathcal{O} = \Delta\, \mathcal{O}\,.}$$

- E.g. one-loop su(2) subsector is just the Heisenberg spin chain

$$\mathcal{D}_2 = \sum 1 - P_{x,x+1} \qquad \mathcal{O} = \mathrm{Tr}(ZZWZ \cdots ZWZ)$$

- Focus on a spin chain with su(2|3) symmetry (arises as subsector of full psu(2,2|4) theory). Each spin site can take one of five orientations

$$\chi = \{Z, \phi^1, \phi^2 \mid \psi^1, \psi^2\}$$

- A generic state is a single trace gauge invariant operator and D=gl(1) generator ⊂ su(2|3).

- Hamiltonian can been explicitly constructed in perturbation theory at low orders

$$g^2 = \frac{\lambda}{8\pi^2} = \frac{g_{YM}^2 N}{8\pi^2}$$

- Perturbative calculations confirm the presence of integrability at low orders (and recent four loop calculations (Bern et al, BES) have confirmed conjectures assuming integrability to all orders).
- Hamiltonian at higher loops is long-ranged, the maximum length of interaction being that of the loop order.
- The action is dynamic, the generators can change the number of spin sites e.g. $\delta\psi \sim [\phi, Z]$
- Define vacuum state: $\cancel{\mathrm{Tr}}\,(ZZZ \cdots ZZZZ)$, consider infinitely long and asymptotic states. Remaining excitations are

$$\chi = \{\phi^1, \phi^2 \mid \psi^1, \psi^2\}$$

the algebra which preserves the number of excitations is su(2|2), consists of two su(2)'s, the central charge (the dilatation operator), the susy charges (which can change the site numbers).

- However it is necessary to extend the algebra to include two additional central charges. This is because the only possible (2|2) representations of su(2|2) have D=±1/2 and anomalous dimensions must vary continously with g.
- Additional central charges vanish when evaluated on physical states satisfying the trace cyclicity condition.
- Using only the symmetry algebra one can find the one impurity eigenvalues of the dilatation operator (for asymptotic states) up to a single arbitrary function of the coupling

$$\frac{1}{2}\sqrt{1+f(g)\sin^2\frac{p}{2}}$$

- To include the effects of interactions between states we use the S-matrix which describes two particle permutations

$$\mathcal{S}_{kl}|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\rangle \mapsto *|\ldots\mathcal{X}''_l\mathcal{X}'''_k\ldots\rangle$$

- The Bethe equations for are then simply the boundary conditions on eigenfunctions for the integrable spin chain Hamiltonian.

$$e^{ip_kL} = \prod S(p_k,p_j)$$

Comparison of Spin Chain/String theory

- The requirement that the S-matrix commutes with (centrally extended) symmety algebra

$$\mathfrak{L}_a{}^b|\phi_c\rangle = \delta_c^b|\phi_a\rangle - \tfrac{1}{2}\delta_a^b|\phi_c\rangle\,, \qquad \mathfrak{R}_\alpha{}^\beta|\psi_\gamma\rangle = \delta_\gamma^\beta|\psi_\alpha\rangle - \tfrac{1}{2}\delta_\alpha^\beta|\psi_\gamma\rangle\,,$$
$$\mathfrak{Q}_\alpha{}^b|\phi_c\rangle = a\,\delta_c^b|\psi_\alpha\rangle\,, \qquad \mathfrak{Q}_\alpha{}^b|\psi_\gamma\rangle = b\,\epsilon_{\alpha\beta}\epsilon^{bc}|\phi_c\mathcal{Z}^+\rangle\,,$$
$$\mathfrak{S}_a{}^\beta|\phi_c\rangle = c\,\epsilon_{ab}\epsilon^{\beta\gamma}|\psi_\gamma\mathcal{Z}^-\rangle\,, \qquad \mathfrak{S}_a{}^\beta|\psi_\gamma\rangle = d\,\delta_\gamma^\beta|\phi_a\rangle\,.$$
$$\mathfrak{P}|\chi\rangle = ab\,|\chi\mathcal{Z}^+\rangle \quad , \quad \mathfrak{K}|\chi\rangle = cd\,|\chi\mathcal{Z}^-\rangle$$

fixes all coefficients up to a single overall factor

$$\mathcal{S}_{12}|\phi_1^a\phi_2^b\rangle = A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \tfrac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi_2^\alpha\psi_1^\beta\mathcal{Z}^-\rangle,$$
$$\mathcal{S}_{12}|\psi_1^\alpha\psi_2^\beta\rangle = D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle + \tfrac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi_2^a\phi_1^b\mathcal{Z}^+\rangle,$$
$$\mathcal{S}_{12}|\phi_1^a\psi_2^\beta\rangle = G_{12}|\psi_2^\beta\phi_1^a\rangle + H_{12}|\phi_2^a\psi_1^\beta\rangle,$$
$$\mathcal{S}_{12}|\psi_1^\alpha\phi_2^b\rangle = K_{12}|\psi_2^\alpha\phi_1^b\rangle + L_{12}|\phi_2^b\psi_1^\alpha\rangle.$$

- In this a key step is using $|\mathcal{Z}^\pm\chi\rangle = e^{\mp ip}|\chi\mathcal{Z}^\pm\rangle$ this is the source of the coproduct for the gauge theory which is similar to that of the worldsheet theory but not identical (a separate gauge issue).

For comparison with string theory we will be particular interested in the small momentum limit: to see why, recall the string length was

$$\mathcal{J} = \frac{2\pi}{\sqrt{\lambda}} P_-$$

and we considered the limit of infinite length with our loop parameter being $\frac{2\pi}{\sqrt{\lambda}}$.

The spin chain length is L= J+α M thus we must rescale all dimensional parameters by $\frac{2\pi}{\sqrt{\lambda}}$ in particular $p_{\mathrm{ch}} = \frac{2\pi}{\sqrt{\lambda}} p_{\mathrm{st}}$

Thus we rescale and expand s.t. $p_k \to 0 \qquad \bar{\lambda} \to \infty \qquad \bar{\lambda} p_k^2 = \text{fixed}$

So for example

$$\mathbf{A}^B = S^0_{pp'} \frac{x^+_{p'} - x^-_p}{x^-_{p'} - x^+_p} \qquad \text{with} \qquad S^0_{pp'} = \frac{1 - \frac{1}{x^+_{p'} x^-_p}}{1 - \frac{1}{x^-_{p'} x^+_p}} e^{i\theta(p,p')}$$

where $$x^\pm_p = \frac{\pi e^{\pm \frac{i}{2} p}}{\sqrt{\lambda} \sin \frac{p}{2}} \left(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right)$$

- The complex spectral parameters $x^{\pm}$ provide a useful description of the worldsheet momenta, dispersion relation, S-matrix etc.

$$\epsilon(x^{\pm}) = \frac{\sqrt{\lambda}}{4\pi i}\left(\left(x^{+} - \frac{1}{x^{+}}\right) - \left(x^{-} - \frac{1}{x^{-}}\right)\right)$$

$$p(x^{\pm}) = \frac{1}{i}\log\frac{x^{+}}{x^{-}} \qquad v = \frac{2\pi}{\sqrt{\lambda}}\frac{d\epsilon}{dp} = \frac{x^{+}+x^{-}}{1+x^{+}x^{-}}$$

- The parameters $x^{\pm}$ satisfy the relation

$$\left(x^{+} + \frac{1}{x^{+}}\right) - \left(x^{+} - \frac{1}{x^{-}}\right) = \frac{4i\pi}{\sqrt{\lambda}}$$

and provide a convenient description of various regimes of particular interest in the strong coupling limit

$$\sqrt{\lambda} \gg 1$$

Three Regimes

- Giant Magnon limit: $p = \text{fixed} \ \sqrt{\lambda} \to \infty$

$$\epsilon \sim \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2} \qquad x^+ \sim \frac{1}{x^-}$$

- Plane-wave limit: $k = \sqrt{\lambda} p = \text{fixed} \ , \sqrt{\lambda} \to \infty \ p \to 0$

$$\epsilon \sim \sqrt{n^2 + k^2} \qquad x^+ \sim x^-$$

- Near-flat space limit (interpolating limit): $\lambda^{\frac{1}{4}} p = \text{fixed}$

$$x^+ \sim x^- \sim -1$$

With Θ equal to that of AFS (which reproduces the semiclassical string spectrum) we get the string theory result for const-J (a=0) gauge .

$$\theta(p,p') = \frac{\sqrt{\lambda}}{2\pi}\sum_{r,s=\pm} rs\,\chi(x_p^r, x_{p'}^s) \qquad \chi(x,y) = (x-y)\left[\frac{1}{xy}+\left(1-\frac{1}{xy}\right)\ln\left(1-\frac{1}{xy}\right)\right]$$

The resulting gauge theory answer is

$$A(p,p') = \frac{1}{4}\Big[(\varepsilon' p - \varepsilon p')^2 - 2(p-p')(\varepsilon' p - \varepsilon p') + (p-p')^2\Big]$$

and the corresponding string answer is

$$A^s(p,p') = \frac{1}{4}\left((1-2a)(\epsilon p' - \epsilon' p)^2 + (p-p')^2\right)$$

There are additional terms due to differences (M) in definition of spin chain/worldsheet length which drop out when we calculate physical quantities. (An appealing simple alternative is to choose S^0=exp(i(p-p')) and we get the a=1/2 result)

We find similar results for all ten coefficients and indeed we can tensor two copies to find the full S-matrix for all physical fields. Just as for the previous case it should be possible to extend this S-matrix to all orders to include worldsheet loop effects.

Finally the spin-chain S-matrix, in a similar fashion to the string S-matrix, does not satisfy the naïve YBE but rather one has to include additional phases due to the $Z^\pm$.

Some further comments

- As we mentioned the spin-chain S-matrix is determined up to the overall phase by the algebra and the action of charges on asymptotic states.
- On the world sheet we don't expect the coproduct to take quantum corrections:
  - –Lagrangian only involves derivatives of x^- and so is local function of transverse fields to any finite order in perturbation theory. Hence we don't expect any renormalization of the currents to introduce any additional factors of x^-.
  - –the x^- field is the only field with non-trivial boundary conditions

$$x^-(-\infty) - x^-(+\infty) = p_{\mathrm{ws}}$$

  and given that in a massive theory we don't expect q. fluctuations to effect long range physics we don't expect the action of x- will be unchanged. In essence we don't expect that the coproduct will receive quantum corrections and so we can expect that we can now make the same argument as on the gauge theory and determine the S-matrix up to an overall phase.

  To understand how integrability is realised we can study the invariance of the higher non-local charges (or even just the first bi-local one) (C.f. two loop Yangian symmetry for spin chains Agarwal&Rajeev, Zwiebel).

Near-Flat Space Model

Let us consider the "near-flat space" limit of Maldacena & Swanson:

Start with a solution: $\dot{t} = 1, \ \dot{\phi} = 1$ and all other fields zero. We now perform a world-sheet boost and expand in fluctuations

$$t = \lambda^{\frac{1}{4}}\sigma^{+} + \frac{\tau}{\lambda^{\frac{1}{4}}} \qquad \phi = \lambda^{\frac{1}{4}}\sigma^{+} + \frac{\chi}{\lambda^{\frac{1}{4}}}$$

$$\vec{\theta} = \frac{\vec{y}}{\lambda^{\frac{1}{4}}} \qquad \vec{\rho} = \frac{\vec{z}}{\lambda^{\frac{1}{4}}}$$

we also expand the fermions in powers of fluctuations

$$\Theta^{1} = \frac{\psi_{-}}{\lambda^{\frac{1}{8}}}, \ \Theta^{2} = \frac{\psi_{+}}{\lambda^{\frac{3}{8}}}$$

and finally take the limit $\lambda \to \infty$

- The resulting Lagrangian is

$$
\begin{aligned}
L \;=\; & \frac{1}{2}\left((\partial z)^2 - z^2 + (\partial y)^2 - y^2\right) - 2i\left(\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_- + \psi_+\Pi\psi_-\right) \\
& +\frac{1}{2}\left[\left(y^2 - z^2\right)\left((\partial_- z)^2 + (\partial_- y)^2\right)\right] - \frac{1}{48}\left[\left(\psi_-\gamma^{jk}\psi_-\right)\left(\psi_-\gamma^{jk}\psi_-\right) - \left(\psi_-\gamma^{j'k'}\psi_-\right)\left(\psi_-\gamma^{j'k'}\psi_-\right)\right] \\
& +\frac{i}{2}\left[\left(z^2 - y^2\right)\left(\psi_-\partial_-\psi_-\right) + \psi_-\left(\partial_- z_j\gamma^j + \partial_- y_{j'}\gamma^{j'}\right)\left(z_k\gamma^k - y_{k'}\gamma^{k'}\right)\psi_-\right].
\end{aligned}
$$

- This can also be reached by taking the appropriate limit (plus field redefinition) of the near-plane wave Lagrangian. In this case we can see that the gauge dependence (i.e. the a-dependence) drops out. Similarly the coproduct becomes trivial and the non-locality drops out. However the interactions are non-trivial.
- However the above action is exact in this limit and we can conjecture that the quantum corrections calculated with this model correspond to the near-flat space limit of the quantum corrections for the full model.
- It is convenient to rescale the fields to put a $1/\sqrt{\lambda}$ in front of the interactions. However this dependence is fake and the Lagrangian only depends on two parameters. In fact if we make a Lorentz transformation plus a rescaling of the coupling we find that the action is invariant.

- The corresponding limit in terms of the spectral parameters is

$$x^{\pm}(p) \to -1 - \frac{1}{p} \pm \frac{i\pi}{\sqrt{\lambda}} p$$

- In this limit the dispersion relation becomes

$$\epsilon = \sqrt{1 + \frac{\sqrt{\lambda}}{\pi} \sin^2 \frac{p\pi}{\sqrt{\lambda}}} \to \sqrt{1 + p_-^2 - \frac{\pi^2 p_-^4}{3\lambda}}$$

with the first correction from the sine function occurring at two-loops.

- The exact S-matrix with the BHL/BES phase in this limit can be written in terms of the rapidity difference $\theta = \theta_1 - \theta_2$ and a momentum dependent coupling $\tilde{\gamma} = \frac{\pi}{\sqrt{\lambda}} p_1 p_2$

$$\mathbb{S} = \frac{e^{\frac{4i\tilde{\gamma}^2}{\pi} \frac{1-\theta \coth \theta}{\sinh \theta}}}{1 + \tilde{\gamma}^2 \coth^2 \frac{\theta}{2}} S \otimes S \,,$$

$$\begin{aligned} A &= 1 + i\tilde{\gamma} \tanh \tfrac{\theta}{2}\,, & B &= -E = 2i\tilde{\gamma} \operatorname{csch} \tfrac{\theta}{2}\,, \\ D &= 1 - i\tilde{\gamma} \tanh \tfrac{\theta}{2}\,, & C &= F = i\tilde{\gamma} \operatorname{sech} \tfrac{\theta}{2}\,, \\ G &= 1 + i\tilde{\gamma}\,, & H &= K = i\tilde{\gamma} \operatorname{csch} \tfrac{\theta}{2}\,, \\ L &= 1 - i\tilde{\gamma}\,. \end{aligned}$$

One-loop S-matrix (Klose&Zarembo)

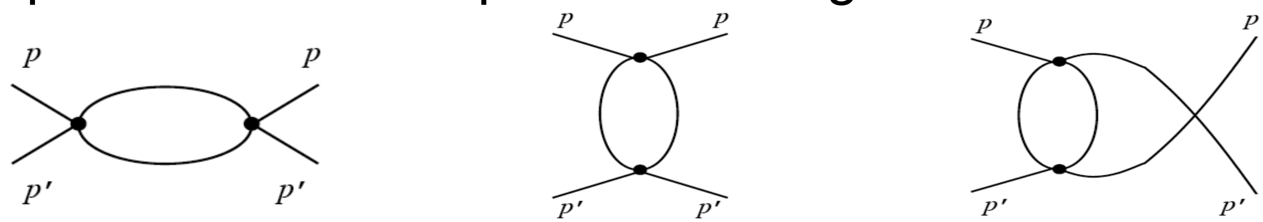
- As an example let us consider the forward scattering of two bosons in a single su(2|2) sector

$$A : \phi_a \phi'_b \to \phi_a \phi'_b$$

- In this limit the tree-level amplitude and S-matrix element are

$$\mathcal{A}_{\text{tree}} = \frac{4i\pi}{\sqrt{\lambda}}(p'^2_- + p^2_-) \qquad S_{\text{tree}} = 1 + \frac{2i\pi}{\sqrt{\lambda}} \frac{(p'^2 + p^2)}{p'^2 - p^2}$$

- At one-loop we have the simple bubble diagrams

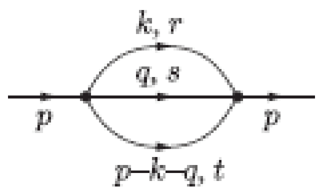


$$\mathcal{A}_{\text{one-loop}} = \frac{8\pi}{\sqrt{\lambda}} \left[(p^2_- + p'^2_-)(p_- + p'_-)^2 I_{00}(p,p') + 8p^2_- p'^2_- I_{00}(p,p) + (p^2_- + p'^2_-)(p_- - p'_-)^2 I_{00}(p,-p') \right]$$

$$\begin{aligned} I_{00}(p,-p') &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{(q^2 - m^2)\left[(q + p' - p)^2 - m^2\right]} \\ &= \frac{i}{2\pi m^2} \frac{p_- p'_-}{p'^2_- - p^2_-} \log \frac{p'_-}{p_-} \end{aligned}$$

Two-loop propagator

- At two-loops there is a non-trivial correction to the propagator. It is necessary to calculate this as it gives corrections to both the dispersion relation and to the wavefunction renormalisation. The relevant diagram is the sunset



$$= \frac{64 i \pi^2 p_-^4}{3\lambda} I_{000}(p_-)$$

$$I_{000}(p_-) = \int \frac{d^2q\ d^2k}{(2\pi)^4} \frac{1}{(k^2-m^2)(q^2-m^2)\left[(p-k-q)^2-m^2\right]}$$

which contributes on-shell $I_{000}(\mathrm{p}^2 = m^2) = \frac{1}{64m^2}$.

We can read off the mass-shift and wavefunction renormalization

$$\frac{i}{p^2 - m^2 + \frac{64 i \pi^2 p_-^4}{3\lambda} I_{000}(p_-)} \equiv \frac{iZ(p_-)}{2p_+ - \Sigma(p_-)} + finite$$

Note: We are considering time evolution in the σ^+ direction so that $2p_+$ is the appropriate "energy".

Two-loop S-matrix

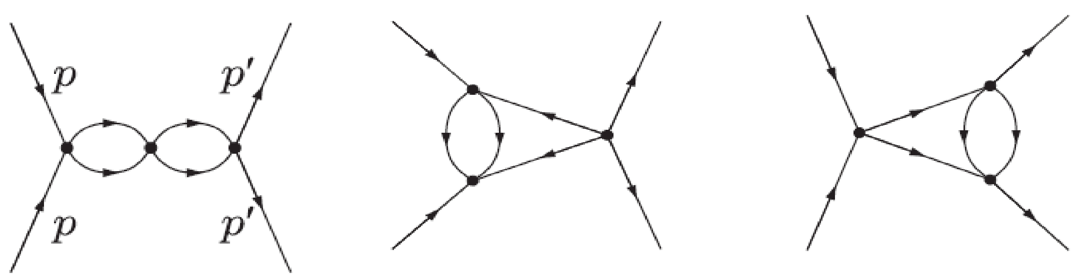
- Important points: Cancellations occur between corrections from δ-function Jacobian

$$\delta(P^\mu_{\rm out} - P^\mu_{\rm in}) = \frac{1}{2}\left(\frac{dp'_+}{dp'_-} - \frac{dp_+}{dp_-}\right)\delta(p_- - q_-)\delta(p'_- - q'_-)$$

where $p_+(p_-) = \frac{1}{2}\Sigma(p_-)$, the wavefunction renormalisation, Z(p_), and the two-loop amplitutde A(p). The full S-matrix is

$$\mathbb{S} = \mathbb{1} + \frac{1}{2}\left(\frac{dp'_+}{dp'_-} - \frac{dp_+}{dp_-}\right)^{-1} Z(p_-)Z(p_-)\mathcal{A}$$

- The relevant diagrams are the double bubble and the wineglass diagram



- Again consider the element for bosonic forward scattering in a su(2|2) sector

$$\mathcal{A}_{A_{\rm two-loop}} = -\frac{16 i\pi^3}{\lambda^{\frac{3}{2}}}(p_-'^2 + p_-^2)\left((p_- + p_-')^4 I_{00}^2(p,p') + (p_- - p_-')^4 I_{00}^2(p,-p')\right.$$
$$\left. -8\,\mathcal{U}_{w.g.} - 8\,\mathcal{S}_{w.g.} + 16\left(p_-^2 \mathcal{T}(p)_{w.g.} + p_-'^2 \mathcal{T}(p')_{w.g.}\right)\right)$$

$$\begin{aligned}
\mathcal{U}_{w.g.} &= 4p_-^2 p_-'^2 W_{00}(p,p') - 8pp'(p+p')W_{10}(p,p') \\
&\quad +(p_-^2 + 6p_-p_-' + p_-'^2)W_{20}(p,p') + ((p_-^2 + 6p_-p_-' + p_-'^2)W_{11}(p,p') \\
\mathcal{S}_{w.g.} &= 4p_-^2 p_-'^2 W_{00}(p,-p') + 8pp'(p-p')W_{10}(p,-p') \\
&\quad +(p_-^2 - 6p_-p_-' + p_-'^2)W_{20}(p,-p') + ((p_-^2 - 6p_-p_-' + p_-'^2)W_{11}(p,-p')
\end{aligned}$$

$$\mathcal{T}(p)_{w.g.} = W_{00}(p,p) - 4p_- W_{10}(p,p) + 2W_{20}(p,p) + 2W_{11}(p,p)$$

$$W_{nm}(p,p') = \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2}\frac{(l_-)^n(k_-)^m}{(k^2-m^2)(l^2-m^2)((k+l-p)^2-m^2)((k+l-p')^2-m^2)}$$

- We can perform the integrals analytically and get significant simplifications.

$$A = \frac{2i\pi^3}{\lambda^{\frac{3}{2}}} p_-^3 p_-'^3 \left[\frac{(p_-^3 + p_-^2 p_-' + p_- p_-'^2 + p_-'^3)}{(p_-' - p_-)^3} + \frac{8 p_- p_-' (p_-^2 + p_-'^2)}{(p_-'^2 - p_-^2)^2} \left(1 - \frac{p_-'^2 + p_-^2}{p_-'^2 - p_-^2} \ln \frac{p_-'}{p_-} \right) \right]$$

- Notably
  - Double logarithms cancel
  - No double poles (c.f. Giant magnon limit where double poles correspond to magnon scattering with light intermediate states going on-shell).
  - Agrees with near-flat space expansion of BHL/BES phase at two-loops
  - Can calculate all elements for an complete su(2|2) sector and show that the appropriate symmetries are present in this limit.

Quantum Corrections in Near-plane wave limit

- Now we wish to calculate the one loop corrections to the four point function and for this we will need the full Lagrangian incl. all fermions and other bosons.
- The full light-cone bosonic Lagrangian to quartic order in fields is

$$L_2 = \frac{1}{2}\left[(\partial_\tau z)^2 - (\partial_\sigma z)^2 + (\partial_\tau y)^2 - (\partial_\sigma y)^2 - p_-^2(z^2+y^2)\right]$$

$$L_4 = \frac{1}{2}y^2(\partial_\sigma y)^2 - \frac{1}{2}z^2(\partial_\sigma z)^2 + \frac{1}{4}y^2\left[(\partial_\sigma z)^2 + (\partial_\tau z)^2\right] - \frac{1}{4}z^2\left[(\partial_\sigma y)^2 + (\partial_\tau y)^2\right]$$

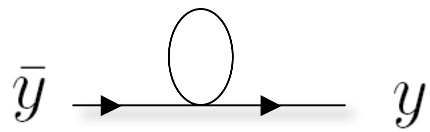
while the terms quadratic in fermions are

$$L_2 = \frac{-ip_-}{2}\left(\bar{S}\rho^\alpha\partial_\alpha S - ip_-\bar{S}\Pi S\right)$$

$$\begin{aligned} L_4 = & -\frac{i}{8}(p_-(\bar{S}\rho^0\partial_\tau S - \bar{S}\rho^1\partial_\sigma S)y^2 + i\bar{S}\Pi S((\partial_\sigma y)^2 - (\partial_\tau y)^2)) \\ & -\frac{ip_-}{16}(\bar{S}\rho^0\Gamma^{ij}S(\partial_\tau y^i\, y^j) - 3\bar{S}\rho^1\Gamma^{ij}S(\partial_\sigma y^i\, y^j)) + \frac{1}{4}\bar{S}\rho^0\rho^1\Gamma^{ij}S\partial_\sigma y^i\partial_\tau y^j \end{aligned}$$

with the fermionic fields S being eight component Majoranna spinors and the ρ's are two dimensional Dirac matrices. Also need all terms with six fields but expressions are a little unwieldy.

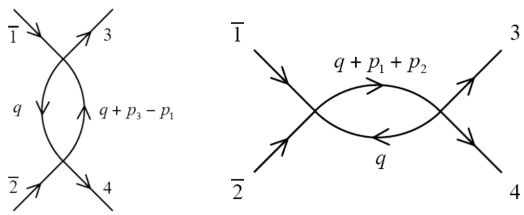
- We can now calculate the one-loop correction to the bosonic two point function and combining contributions from the bosons and fermions we find

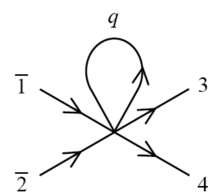


$$G^{(2)}(\bar{1},2) = (\omega_1\omega_2 - n_1 n_2 - p_-^2)\int \frac{d^{2-2\epsilon}q}{(2\pi)}\frac{1}{q^2+p_-^2} + finite$$

- If we now impose energy/momentum conservation and the quadratic equations of motion the divergent term goes away.
- This occurs as a non-trivial combination of the bosonic and fermionic contributions and after dropping integrals of the type $\int d^{2-2\epsilon}q \times 1$.
- Even off-shell such a divergence can be removed by a redefinition of the fields.

- We now calculate the four point function which receives contributions from terms involving
  - two quartic vertices and vertices with six fields





  - If we calculate in the c.o.m frame and use the dispersion relation we find that

$$G^{(4)}(p,-p,p',-p') \sim \frac{1}{p_-^2}(\omega^2 n_{p'}^2 + \omega_{p'}^2 n^2) \int \frac{d^{2-2\epsilon}q}{(2\pi)} \frac{1}{q^2+p_-^2} + finite$$

  - which does not vanish nor does it seem to be removable by renormalizing the fields.
- It is perhaps most illuminatingly written as a divergent contribution to the effective action in coordinate space and for the full SO(4) vectors (and after using the equations of motion to simplify the expressions)

$$\left[-\frac{1}{p_-^2}(\partial_\tau y \cdot \partial_\sigma y)^2 + \frac{1}{p_-^2}(\partial_\sigma y)^2(\partial_\tau y)^2 - 2y^2(\partial_\sigma y)^2\right] \int \frac{d^{2-2\epsilon}q}{q^2+p_-^2} + \text{finite}$$

- Several comments are in order
  - In light-cone gauge with a curved world-sheet metric the ghosts do not decouple and so we must include their contribution at one loop. There is no contribution to the two point function and unfortunately they do not remove the divergences from the four point function.
  - The finite part of the four point function does not vanish when one considers zero-mode states, these excitations are BPS and so should remain free. A similar problem arises in the calculation of near-BMN energies and is remedied by making a field redefinition for the fermions. In fact we are allowed to make arbitrary field redefinitions e.g.

$$S \to S + M(y)S, \qquad M(y) = A(y) + \rho^0 B(y) + \rho^1 C(y) + \rho^0 \rho^1 D(y)$$

    and we expect that a similar redefinition will remove the zero-mode interactions in this case. We can further ask if a field redefinition will remove the divergences unfortunately this does not seem to be the case. However it is possible that a more general redefinition may work

  -- There is also the issue of world-sheet diffeomorphism invariance. We might expect that a different choice of gauge would remove these divergences. Certainly trying a different light-cone gauge where J is uniformly distributed rather than P_- does not seem to fix these issues however perhaps there is a more general transformation that would.

Summary

- It is of interest to find as much information about the string S-matrix as possible and here we have described some partial results:
  - Studying the scattering of excitations in the physical light-cone like gauges: tree level results seem to make sense, once we include novel Hopf algebra structure and we get good agreement with the corresponding gauge theory calculations.
  - Calculate higher-loop quantum corrections to the S-matrix in the near-flat space limit. Here we find agreement at two-loops with the conjectured exact asymptotic S-matrix.
  - For the full theory in the near-plane wave limit at one-loop we find divergences which are difficult to interpret and suggest an alternative approach is possibly needed.
  - Future work:
    - Remaining issues regarding Hopf algebra structure, YBE, non-local charges. Similar continuouso models with novel Hopf alg. su(2|3) LL?
    - Find a redefinition to remove divergences in l.c. gauge or find a sensible interpretation.
    - Higher loops in the nfs-model? Presumably integrable to all orders: Exactly solvable?