

Fermionic Superstring Loop Amplitudes in the Pure Spinor Formalism

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Outline

Introduction

Methods for kinematic factor evaluation

Superfield components

Integration in pure spinor superspace

Reduction to kinematic bases

Application to one- and two-loop amplitudes

One loop

Two loops

Discussion

Pure spinor quantisation of the superstring

[Berkovits (2000)]

Problems with superstring quantisation:

- RNS: Spacetime susy only after summation over spin structures
- GS: Need to go to light cone gauge

Flat-space action:

$$S = \int d^2z \left(-\frac{1}{2} \partial x^m \bar{\partial} x_m - p_\alpha \bar{\partial} \theta^\alpha - \tilde{p}_{\bar{\alpha}} \bar{\partial} \tilde{\theta}^{\bar{\alpha}} + w_\alpha \bar{\partial} \lambda^\alpha + \tilde{w}_{\bar{\alpha}} \partial \tilde{\lambda}^{\bar{\alpha}} \right)$$

- Bosonic ghost λ subject to pure spinor constraint:

$$\lambda \gamma^m \lambda = 0$$

Critical in ten dimensions.

- BRST operator $Q = \int \lambda^\alpha d_\alpha$: cohomology reproduces superstring spectrum
- Tree amplitudes agree with RNS [Berkovits & Vallilo (2000)]
- Relate to GS formalism in semi-lightcone gauge

Pure spinor loop amplitudes

Multiloop amplitudes constructed in pure spinor formulation, and used to prove vanishing theorems [Berkovits (2004)]. Massless multiloop amplitude: Need at least four external states. One-loop amplitude:

$$\mathcal{A} = K\bar{K} \int \frac{d^2\tau}{(\text{Im } \tau)^5} \int d^2z_2 \int d^2z_3 \int d^2z_4 \prod_{i < j} G(z_i, z_j)^{k_i \cdot k_j}$$

Kinematic factors expressed as “pure spinor superspace integrals”:

$$\begin{aligned} K_{1\text{-loop}} &= \langle (\lambda A)(\lambda \gamma^m W)(\lambda \gamma^n W) \mathcal{F}_{mn} \rangle \\ K_{2\text{-loop}} &= \langle (\lambda \gamma^{mnpqr} \lambda)(\lambda \gamma^s W) \mathcal{F}_{mn} \mathcal{F}_{pq} \mathcal{F}_{rs} \rangle \end{aligned}$$

Covariant zero mode integration: $\langle \lambda^3 \theta^5 \rangle = 1$. Superfields:

$$A_\alpha \sim (\gamma^a \theta)_\alpha \zeta_a + (u \gamma^a \theta)(\gamma_a \theta)_\alpha + \dots$$

and $A_\alpha \xrightarrow{D_\beta} A_m \xrightarrow{D_\beta} W^\alpha \xrightarrow{D_\beta} \mathcal{F}_{mn}$, where $D_\alpha = \partial_\alpha + \frac{1}{2} k_a (\gamma^a \theta)_\alpha$.

Bosonic amplitudes agree with RNS [Berkovits & Mafra, Mafra (2005)]

The problems addressed in this talk

Evaluation of pure spinor superspace integrals in kinematic factors:

$$K_{\text{2-loop}} = \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} W(\theta)) \mathcal{F}_{mn}(\theta) \mathcal{F}_{pq}(\theta) \mathcal{F}_{rs}(\theta) \rangle$$

↓ Expand superfields, saturate θ^5

$$\cdots + k_a^2 k_m^2 k_p^3 k_r^4 \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} u_1) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma_b u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle + \cdots$$

↓ Evaluate correlator

$$\begin{aligned} \langle \cdots \rangle &= (u_1 \gamma^{mpr} u_2) (u_3 \gamma^a u_4) + (u_1 \gamma^{ai_1 i_2} u_2) (u_3 \gamma^{mpr} {}_{i_1 i_2} u_4) + \cdots \\ &\quad + \varepsilon^{ampr} {}_{i_1 \dots i_6} (u_1 \gamma^{i_3 \dots i_9} u_2) (u_3 \gamma^{i_1 i_2} {}_{i_7 i_8 i_9} u_4) + \cdots \end{aligned}$$

↓ Reduce to kinematic basis (Use on-shell identities)

$$K = s_{12} [(u_1 \not{k}_3 u_2) (u_3 \not{k}_1 u_4) - (u_1 \not{k}_2 u_3) (u_2 \not{k}_1 u_4) + (u_1 \not{k}_2 u_4) (u_2 \not{k}_1 u_3)]$$

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Superfield recursion relations

[Ooguri et al. (2000), Grassi & Tamassia (2004), Policastro & Tsimpis (2006)]

$$A_\alpha(x, \theta) = 0 + \frac{1}{2}(\theta\gamma^a)_\alpha \zeta_a + \frac{1}{3}(\theta\gamma^a)_\alpha (\theta\gamma_a u) + ? \quad A_m, W^\alpha, \mathcal{F}_{mn} = ?$$

Choose a gauge $\theta^\alpha A_\alpha = 0$. On-shell identities

$$2D_{(\alpha} A_{\beta)} = \gamma_{\alpha\beta}^m A_m, \quad D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_{\alpha}{}^\beta \mathcal{F}_{mn}$$

lead to recursion relations

$$A_\alpha^{(n)} = \frac{1}{n+1}(\gamma^m \theta)_\alpha A_m^{(n-1)}$$

$$A_m^{(n)} = \frac{1}{n}(\theta\gamma_m W^{(n-1)})$$

$$W^{\alpha(n)} = -\frac{1}{2n}(\gamma^{mn}\theta)^\alpha \partial_m A_n^{(n-1)}$$

where $f^{(n)} = \frac{1}{n!} \theta^{\alpha_n} \cdots \theta^{\alpha_1} (D_{\alpha_1} \cdots D_{\alpha_n} f)$. Can solve: obtain e.g.

$A_m^{(n)}$ from ζ, u via derivative operator $\mathcal{O}_m q = \frac{1}{2}(\theta\gamma_m^{qp}\theta)\partial_p$:

$$A_m^{(2k)} = \frac{1}{(2k)!} [\mathcal{O}^k]_m{}^q \zeta_q, \quad A_m^{(2k+1)} = \frac{1}{(2k+1)!} [\mathcal{O}^k]_m{}^q (\theta\gamma_q u).$$

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Pure spinor integrals: Tensorial formulae

Recall: Lorentz invariant integration

$$\bar{T}^{\alpha\beta\gamma,\delta_1\dots\delta_5} \equiv \langle \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\delta_1} \dots \theta^{\delta_5} \rangle = \bar{T}^{(\alpha\beta\gamma),[\delta_1\dots\delta_5]}$$

with normalisation $\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1$.

Uniquely determined by symmetry

$$\lambda^{(\alpha} \lambda^\beta \lambda^\gamma) : \quad \text{Sym}^3 S^+ = [00003] \oplus [10001]$$

$$\theta^{[\delta_1} \dots \theta^{\delta_5]} : \quad \text{Alt}^5 S^+ = [00030] \oplus [11010].$$

Can therefore derive tensorial expressions such as

$$\langle (\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_a\theta)(\lambda\gamma_b\theta)(\theta\gamma_{cde}\theta) \rangle = -\frac{1}{42} (\delta_{abcde}^{mnpqr} + \frac{1}{5!} \varepsilon^{mnpqr} \gamma_{abcde})$$

Generally, by Fierzing, reduce to three basic correlators:

$$\langle (\lambda\gamma^{[5]}\lambda)(\lambda\{\gamma^{[1]} \text{ or } \gamma^{[3]} \text{ or } \gamma^{[5]}\}\theta)(\theta\gamma^{[3]}\theta)(\theta\gamma^{[3]}\theta) \rangle.$$

Tensorial formulae: example application

Example from two-loop four-fermion amplitudes:

$$\tilde{F} = \langle (\lambda \gamma^{mnpq[r]} \lambda) (\lambda \gamma^{s]} u_1) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma_b u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle.$$

After Fierz transformations:

$$\begin{aligned} \tilde{F} = & \left(\frac{1}{16} \langle (\lambda \gamma^{mnpq[r]} \lambda) (\lambda \gamma^c \theta) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma^{jkl} \theta) \rangle (u_1 \gamma^{[s]} \gamma_c \gamma_b u_2) \right. \\ & + \frac{1}{3! \cdot 16} \langle (\lambda \gamma^{mnpq[r]} \lambda) (\lambda \gamma^{cde} \theta) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma^{jkl} \theta) \rangle (u_1 \gamma^{[s]} \gamma_{cde} \gamma_b u_2) \\ & + \frac{1}{2 \cdot 5! \cdot 16} \langle (\lambda \gamma^{mnpq[r]} \lambda) (\lambda \gamma^{cdefg} \theta) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma^{jkl} \theta) \rangle (u_1 \gamma^{[s]} \gamma_{cdefg} \gamma_b u_2) \\ & \left. \times \frac{1}{3! \cdot 16} (u_3 \gamma_q \gamma_{jkl} \gamma_s u_4), \right. \end{aligned}$$

Use correlator formulae such as

$$\begin{aligned} \langle (\lambda \gamma^{mnpqr} \lambda) (\lambda \gamma^{abcde} \theta) (\theta \gamma^{fgh} \theta) (\theta \gamma^{jkl} \theta) \rangle &= \frac{16}{7} \left(\delta_{\bar{m}\bar{n}\bar{p}\bar{q}\bar{r}}^{mnpqr} + \frac{1}{5!} \varepsilon^{mnpqr} \bar{m}\bar{n}\bar{p}\bar{q}\bar{r} \right) \\ &\times \left[\delta_{abc}^{\bar{m}\bar{n}\bar{p}} \delta_j^d \delta_g^d \delta_k^{\bar{q}} (-\delta_h^e \delta_l^{\bar{r}} + 2\delta_l^e \delta_h^{\bar{r}}) + \delta_{ab}^{\bar{m}\bar{n}} \delta_{fg}^{cd} \delta_{jk}^{\bar{p}\bar{q}} (\delta_h^e \delta_l^{\bar{r}} - 3\delta_l^e \delta_h^{\bar{r}}) \right]_{[abcde][fgh][jkl](fgh \leftrightarrow jkl)} \end{aligned}$$

⇒ Fairly complicated

Reduction to traces and bilinears

Covariant expression for correlator, using gamma matrices:

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\delta_1} \dots \theta^{\delta_5} \rangle = N^{-1} [(\gamma^m)^{\alpha\delta_1} (\gamma^n)^{\beta\delta_2} (\gamma^p)^{\gamma\delta_3} (\gamma_{mnp})^{\delta_4\delta_5}]_{(\alpha\beta\gamma)[\delta_1\dots\delta_5]}$$

Normalisation constant N fixed by

$$\begin{aligned} N \times & \left\langle (\lambda \gamma_x \theta)(\lambda \gamma_y \theta)(\lambda \gamma_z \theta)(\theta \gamma^{xyz} \theta) \right\rangle \\ &= [(\gamma^m)^{\alpha\delta_1} (\gamma^n)^{\beta\delta_2} (\gamma^p)^{\gamma\delta_3} (\gamma_{mnp})^{\delta_4\delta_5}]_{(\alpha\beta\gamma)[\delta_1\dots\delta_5]} \\ &\quad \times (\gamma_x)_{\alpha\delta_1} (\gamma_y)_{\beta\delta_2} (\gamma_z)_{\gamma\delta_3} (\gamma^{xyz})_{\delta_4\delta_5} \\ &= -\frac{1}{60} \text{Tr}(\gamma_x \gamma^m) \text{Tr}(\gamma_y \gamma^n) \text{Tr}(\gamma_z \gamma^p) \text{Tr}(\gamma^{xyz} \gamma_{pnm}) + \dots \\ &\quad -\frac{1}{60} \text{Tr}(\gamma_z \gamma_{pnm}) \gamma^{zyx} \gamma^n \gamma_x \gamma^m \gamma_y \gamma^p) \quad (60 \text{ terms}) \\ &= 5160960. \end{aligned}$$

Use computer algebra to

- carry out spinor index symmetrisations
- simplify γ products and traces (e.g. Mathematica with GAMMA)

Reduction to traces and bilinears (2)

Allows for easier evaluation of fermionic correlators:

$$\begin{aligned}
 & N \langle (\lambda \gamma^{mnpq[r]} \lambda) (\lambda \gamma^s u_1) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma_b u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle \\
 &= \frac{1}{60} \text{Tr}(\gamma_x \gamma^{ab} {}_n \gamma_y \gamma^{mnpq[r]})(u_3 \gamma_q \gamma^{xyz} \gamma_s u_4) (u_1 \gamma^{|s|} \gamma_z \gamma_b u_2) + \dots \\
 &\quad - \frac{1}{30} (u_2 \gamma_b \gamma^{xyz} \gamma_q u_3) (u_1 \gamma_s \gamma_y \gamma^{ab} {}_n \gamma_x \gamma^{mnpq[r]} \gamma_z \gamma^{|s|} u_4), \quad (24 \text{ terms})
 \end{aligned}$$

Simpler because:

- no Fierzing necessary
- fewer open indices \Rightarrow fewer ε terms
- easier to put on computer

Component approach

Can also use gamma matrix expression

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\delta_1} \dots \theta^{\delta_5} \rangle = N^{-1} [(\gamma^m)^{\alpha\delta_1} (\gamma^n)^{\beta\delta_2} (\gamma^p)^{\gamma\delta_3} (\gamma_{mnp})^{\delta_4\delta_5}]_{(\alpha\beta\gamma)[\delta_1\dots\delta_5]}$$

to compute correlators of particular spinor components.

Example: Check coefficients in

$$\begin{aligned} t_{10}^{mmn m_1 n_1 \dots m_4 n_4} &\equiv \langle (\lambda \gamma^p \gamma^{m_1 n_1} \theta) (\lambda \gamma^q \gamma^{m_2 n_2} \theta) (\lambda \gamma^r \gamma^{m_3 n_3} \theta) (\theta \gamma^m \gamma^n \gamma_{pqr} \gamma^{m_4 n_4} \theta) \rangle \\ &= -\frac{2}{45} (\eta^{mn} t_8^{m_1 n_1 \dots m_4 n_4} - \frac{1}{2} \varepsilon^{mn m_1 n_1 \dots m_4 n_4}) \end{aligned}$$

Choose particular index values. Expand integrand and evaluate correlators on all monomials (here, about 10^5) :

$$\begin{aligned} &\langle (\lambda \gamma^p \gamma^{12} \theta) (\lambda \gamma^q \gamma^{21} \theta) (\lambda \gamma^r \gamma^{34} \theta) (\theta \gamma^0 \gamma^0 \gamma_{pqr} \gamma^{43} \theta) \rangle \\ &= \langle 12 \lambda^1 \lambda^1 \lambda^1 \theta^1 \theta^9 \theta^{10} \theta^{11} \theta^{12} + \dots + 12 \lambda^{16} \lambda^{16} \lambda^{16} \theta^5 \theta^6 \theta^7 \theta^8 \theta^{16} \rangle = \frac{1}{45}. \end{aligned}$$

Probably unsuitable for more complex applications.

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Purely bosonic expressions

Easy to algorithmically deal with bosonic on-shell identities:

- momentum conservation $\sum_i k_i = 0$ and masslessness $k_i^2 = 0$:
 - eliminate one momentum (e.g. k_4) and set all k_i^2 to zero
 - eliminate one quadratic invariant (e.g. $s_{23} \rightarrow -s_{12} - s_{13}$)
- equation of motion for polarisation vector $k_i \cdot \zeta_i = 0$:
 - set all $k_i \cdot \zeta_i$ to zero
 - replace one extra $k \cdot \zeta$, e.g. $k_3 \cdot \zeta_4 \rightarrow (-k_1 - k_2) \cdot \zeta_4$

For gauge invariant expressions: start with $F_i^{ab} = 2k_i^{[a}\zeta_i^{b]}$, e.g.

- gauge invariant $k^4 \zeta_1 \zeta_2 \zeta_3 \zeta_4$ scalars:

$$\text{Tr}(F_1 F_2 F_3 F_4)$$

$$\text{Tr}(F_1 F_2) \text{Tr}(F_3 F_4)$$

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- gauge invariant $k^6 \zeta_1 \zeta_2 \zeta_3 \zeta_4$ scalars: additionally terms like

$$k_3 \cdot F_1 \cdot F_2 \cdot k_3 \text{Tr}(F_3 F_4), \quad k_4 \cdot F_1 \cdot F_3 \cdot k_2 \text{Tr}(F_2 F_4)$$

Fermions: Fierz identities

Spinors – more complicated.

Example question: which scalars from $u_1 u_2 u_3 u_4$?

- start from

$$\begin{aligned} T_1(1234) &= (u_1 \gamma^a u_2)(u_3 \gamma_a u_4), \\ T_3(1234) &= (u_1 \gamma^{abc} u_2)(u_3 \gamma_{abc} u_4). \end{aligned}$$

- Fierz transformations $\Rightarrow T_3(1234) = -12T_1(1234) - 24T_1(1324)$
- $(\gamma_a)_{(\alpha\beta} (\gamma^a)_{\gamma)\delta} = 0 \Rightarrow T_1(1234) + T_1(1324) + T_1(1423) = 0$
- leaves two scalars $T_1(1234)$ and $T_1(1324)$
- agrees with rep theory, $(S^+)^{\otimes 4} = 2 \cdot \mathbf{1} + \dots$

Easy to computerise in matrix representation for γ^a :

Reduce to independent monomials $u_1^\alpha u_2^\beta u_3^\gamma u_4^\delta$, e.g.

$$T_1(1234) = 2u_1^9 u_2^9 u_3^1 u_4^1 + 2u_1^{10} u_2^{10} u_3^1 u_4^1 + \dots$$

Fermions: Dirac equation

In one- and two-loop amplitudes: need $(k^2 \text{ or } k^4) \times u_1 u_2 u_3 u_4$.

- Spinors subject to $\not{k}_i u_i = 0$
- Useful observation: only need one k into $(u_i \gamma^{[n]} u_j)$ bilinears.
Thus consider $(k^2 \text{ or } k^4) \times (u_1 u_2 u_3 u_4 \text{ scalar or 2-tensor})$
- Computer treatment: solve Dirac equation with $\text{SO}(8)$ spinors

$$u = \begin{pmatrix} u^s \\ u^c \end{pmatrix} \quad \text{with} \quad \gamma^{1\dots 8} \begin{pmatrix} u^s \\ u^c \end{pmatrix} = \begin{pmatrix} +u^s \\ -u^c \end{pmatrix}.$$

Obtain coupled equations

$$(k_0 + k_9)u^s - (\sigma \cdot k)u^c = 0, \quad (k_0 - k_9)u^c - (\sigma^T \cdot k)u^s = 0$$

with solution $u^s = (\sigma \cdot k)u^c / \sqrt{2}k_+$.

- Everything reduced to independent monomials in $(u_i^c)^{1\dots 8}$

Using particular momenta

This algorithm reduces all scalars containing k_i and u_i to

$$\sum f_{\alpha\beta\gamma\delta}(k_i)(u_1^c)^\alpha(u_2^c)^\beta(u_3^c)^\gamma(u_4^c)^\delta$$

Problem: manifest Lorentz invariance broken by γ^a matrix rep.

\Rightarrow difficult to deal with the $f_{\alpha\beta\gamma\delta}(k_i)$ (e.g. terms proportional to $k_i^2 = 0$)

One solution: substitute sets of particular vectors for k_i , e.g.

$$k_1^\mu = (5, 3, 0, \dots, 0, 4)$$

$$k_2^\mu = (-5, 0, 3, \dots, 4, 0)$$

$$k_3^\mu = (5, -3, 0, \dots, -4, 0)$$

$$k_4^\mu = (-5, 0, -3, \dots, 0, -4)$$

Results: Algorithms to find independent scalars, decomposition algorithms, e.g. for $k^2 u_1 \dots u_4$ scalars,

$$(u_1 k_3 u_2)(u_3 k_1 u_4)$$

$$s_{12} T_1(1234)$$

$$(u_1 k_2 u_3)(u_2 k_1 u_4)$$

$$s_{13} T_1(1234)$$

$$(u_1 k_2 u_4)(u_2 k_1 u_3)$$

$$s_{13} T_1(1324)$$

The problems addressed in this talk

Evaluation of pure spinor superspace integrals in kinematic factors:

$$K_{\text{2-loop}} = \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} W(\theta)) \mathcal{F}_{mn}(\theta) \mathcal{F}_{pq}(\theta) \mathcal{F}_{rs}(\theta) \rangle$$

↓ Expand superfields, saturate θ^5

$$\cdots + k_a^2 k_m^2 k_p^3 k_r^4 \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} u_1) (\theta \gamma_n{}^{ab} \theta) (\theta \gamma_b u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle + \cdots$$

↓ Evaluate correlator

$$\begin{aligned} \langle \cdots \rangle &= (u_1 \gamma^{mpr} u_2) (u_3 \gamma^a u_4) + (u_1 \gamma^{ai_1 i_2} u_2) (u_3 \gamma^{mpr} {}_{i_1 i_2} u_4) + \cdots \\ &\quad + \varepsilon^{ampr} {}_{i_1 \dots i_6} (u_1 \gamma^{i_3 \dots i_9} u_2) (u_3 \gamma^{i_1 i_2} {}_{i_7 i_8 i_9} u_4) + \cdots \end{aligned}$$

↓ Reduce to kinematic basis (Use on-shell identities)

$$K = s_{12} [(u_1 \not{k}_3 u_2) (u_3 \not{k}_1 u_4) - (u_1 \not{k}_2 u_3) (u_2 \not{k}_1 u_4) + (u_1 \not{k}_2 u_4) (u_2 \not{k}_1 u_3)]$$

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One-loop massless four-point amplitude

Derived in pure spinor formalism [Berkovits (2004)]:

$$\mathcal{A} = K \bar{K} \int \frac{d^2\tau}{(\text{Im } \tau)^5} \int d^2 z_2 \int d^2 z_3 \int d^2 z_4 \prod_{i < j} G(z_i, z_j)^{k_i \cdot k_j}$$

Kinematic factor:

$$K = \langle (\lambda A_1)(\lambda \gamma^m W_2)(\lambda \gamma^n W_3) \mathcal{F}_{4,mn} \rangle + (\text{cycl}(234))$$

Saturate θ^5 :

$$X_{ABCD} = \left\langle (\lambda A_1^{(A)})(\lambda \gamma^m W_2^{(B)})(\lambda \gamma^n W_3^{(C)}) \mathcal{F}_{4,mn}^{(D)} \right\rangle$$

with $A + B + C + D = 5$.

Parities of $A, B, C, D \leftrightarrow$ bosonic or fermionic external states

Four bosons

(Review of [Mafra (2005)])

Can evaluate correlator for one labelling, and then symmetrise:

$$\begin{aligned} K_{\text{1-loop}}^{(4B)} &= \langle (\lambda A_1)(\lambda \gamma^m W_2)(\lambda \gamma^n W_3) \mathcal{F}_{4,mn} \rangle|_{4B} + (\text{cycl } (234)) \\ &= X_{3110} + X_{1310} + X_{1130} + X_{1112} + (\text{cycl } (234)) \end{aligned}$$

Result must be a linear combination of single trace $\text{Tr}(F_{(1} F_2 F_3 F_{4)})$ and double trace $\text{Tr}(F_{(1} F_{2)} \text{Tr}(F_3 F_4))$. Pure spinor integrals:

$$\begin{aligned} X_{3110} &= \frac{15}{64} F_{mn}^1 F_{pq}^2 F_{rs}^3 F_{tu}^4 \langle (\lambda \gamma^{[t|} \gamma^{pq} \theta)(\lambda \gamma^{[u]} \gamma^{rs} \theta)(\lambda \gamma_a \theta)(\theta \gamma^{amn} \theta) \rangle, \\ X_{1112} &= \frac{15}{16} k_m^4 \zeta_n^1 F_{pq}^2 F_{rs}^3 F_{tu}^4 \langle (\lambda \gamma^{[m|} \gamma^{pq} \theta)(\lambda \gamma^{[a]} \gamma^{rs} \theta)(\lambda \gamma^n \theta)(\theta \gamma_a^{tu} \theta) \rangle, \\ X_{1310} &= \frac{5}{16} k_m^3 \zeta_n^1 F_{pq}^2 F_{rs}^3 F_{tu}^4 \langle (\lambda \gamma^{[t|} \gamma^{ma} \theta)(\lambda \gamma^{[u]} \gamma^{rs} \theta)(\lambda \gamma^n \theta)(\theta \gamma_a^{pq} \theta) \rangle. \end{aligned}$$

Reduce to traces:

$$\begin{aligned} \tilde{X}_{3110} &= N^{-1} \left[\frac{1}{60} \text{Tr}(\gamma_a \gamma^z) \text{Tr}(\gamma_{xyz} \gamma^{anm}) \text{Tr}(\gamma^x \gamma_{qp} \gamma^{[t|}) \text{Tr}(\gamma^y \gamma_{sr} \gamma^{[u]}) + \dots \right. \\ &\quad \left. \dots + \frac{1}{60} \text{Tr}(\gamma^{[u]} \gamma_{rs} \gamma_{zyx} \gamma_{qp} \gamma^{[t|}) \gamma^x \gamma_a \gamma^y \gamma^{mna} \gamma^z \right] \quad (60 \text{ terms}) \end{aligned}$$

$$= \left(\frac{1}{35} \delta_{rs}^{mp} \delta_{tu}^{nq} - \frac{1}{630} \delta_{tu}^{mn} \delta_{rs}^{pq} - \frac{1}{90} \delta_{rs}^{mn} \delta_{tu}^{pq} + \frac{26}{315} \delta_{pr}^{mn} \delta_{tu}^{qs} \right)_{[mn][pq][rs][tu]} (pq \leftrightarrow rs)$$

Four bosons (2)

Contract with field strengths, momenta and polarisations, and symmetrise:

$$\begin{aligned} X_{3110} + (\text{cycl}(234)) &= \frac{11}{112} \text{Tr}(F_{(1} F_2 F_3 F_{4)}) - \frac{1}{56} \text{Tr}(F_{(1} F_2) \text{Tr}(F_3 F_{4})) \\ X_{1112} + (\text{cycl}(234)) &= \frac{19}{448} \text{Tr}(F_{(1} F_2 F_3 F_{4)}) - \frac{31}{1792} \text{Tr}(F_{(1} F_2) \text{Tr}(F_3 F_{4})) \\ X_{1310} + (\text{cycl}(234)) &= \frac{3}{512} (4 \text{Tr}(F_{(1} F_2 F_3 F_{4)}) - \text{Tr}(F_{(1} F_2) \text{Tr}(F_3 F_{4}))) \end{aligned}$$

Total:

$$K_{\text{1-loop}}^{\text{4B}} = \frac{3}{64} (4 \text{Tr}(F_{(1} F_2 F_3 F_{4)}) - \text{Tr}(F_{(1} F_2) \text{Tr}(F_3 F_{4}))) = \frac{1}{128} t_8 F^4$$

Agrees with Green-Schwarz and RNS results

Four fermions

Could similarly evaluate:

$$K_{\text{1-loop}}^{(4F)} = \langle (\lambda A_1)(\lambda \gamma^m W_2)(\lambda \gamma^n W_3) \mathcal{F}_{4,mn} \rangle|_{4F} + (\text{cycl } (234))$$

Note the symmetry:

- cycl(234) implies complete **antisymmetry** in u_2, u_3, u_4
- on $k^2 u_1 u_2 u_3 u_4$ scalars, [234] is equivalent to [1234]
- there is **only one** antisymmetric $k^2 u_1 u_2 u_3 u_4$ scalar

Thus the result is fixed by symmetry:

$$K_{\text{1-loop}}^{4F} \sim (u_1 k_3 u_2)(u_3 k_1 u_4) - (u_1 k_2 u_3)(u_2 k_1 u_4) + (u_1 k_2 u_4)(u_2 k_1 u_3)$$

Again, agrees with previously known expressions

Two bosons, two fermions

Symmetrisations more tricky now:

$$\begin{aligned} K_{\text{1-loop}}^{\text{2B2F}}(f_1 f_2 b_3 b_4) = & (1 - \pi_{34}) \langle (\lambda A_1^{\text{(even)}})(\lambda \gamma^m W_2^{\text{(even)}})(\lambda \gamma^n W_3^{\text{(odd)}}) \mathcal{F}_{4,mn}^{\text{(even)}} \rangle \\ & + \langle (\lambda A_1^{\text{(even)}})(\lambda \gamma^m W_3^{\text{(odd)}})(\lambda \gamma^n W_4^{\text{(odd)}}) \mathcal{F}_{2,mn}^{\text{(odd)}} \rangle. \end{aligned}$$

First line has wrong [34] symmetry and must give zero. Fermionic expansion:

$$K_{\text{1-loop}}^{\text{2B2F}} = (1 - \pi_{34}) (X_{4010} + X_{2210} + X_{2030} + X_{2012}) + X'_{2111},$$

Typical correlator:

$$X_{4010} = -\frac{1}{60} k_q^1 k_b^3 \zeta_c^3 k_m^4 \zeta_n^4 \langle (\lambda \gamma^a \theta)(\theta \gamma_a^{pq} \theta)(\theta \gamma_p u_1)(\lambda \gamma^{[m} u_2)(\lambda \gamma^{n]} \gamma^{bc} \theta) \rangle$$

Reduce to $u_1 u_2$ bilinears:

$$\begin{aligned} \tilde{X}_{4010} = & \frac{1}{360} \delta_{mn}^{q[b} (u_1 \gamma^{c]} u_2) + \frac{1}{90} \delta_{bc}^{q[m} (u_1 \gamma^{n]} u_2) + \frac{1}{720} \delta_{mn}^{bc} (u_1 \gamma^q u_2) \\ - & \frac{1}{2520} \delta_q^{[m} (u_1 \gamma^{n]} {}^{bc} u_2) - \frac{1}{720} \delta_q^{[b} (u_1 \gamma^{c]} {}^{mn} u_2) - \frac{1}{1260} \delta_{[m}^{[b} (u_1 \gamma_{n]}^{c]q} u_2) + \frac{1}{3360} (u_1 \gamma^{bcmnq} u_2) \end{aligned}$$

Two bosons, two fermions (2)

Contract with k and ζ , and expand in kinematic basis $C_1 \dots C_5$:

$$\begin{aligned}(1 - \pi_{34}) X_{4010} &= \frac{1}{60480} (-1, 12, -12, 1, 2)_{C_1 \dots C_5} \\(1 - \pi_{34}) X_{2210} &= \frac{1}{60480} (31, -30, 30, -1, -2)_{C_1 \dots C_5} \\(1 - \pi_{34}) X_{2030} &= \frac{1}{60480} (0, -21, 21, 0, 0)_{C_1 \dots C_5} \\(1 - \pi_{34}) X_{2012} &= \frac{1}{60480} (-30, 39, -39, 0, 0)_{C_1 \dots C_5} \\X'_{2111} &= \frac{1}{480} (1, 0, 4, -1, 0)_{C_1 \dots C_5}\end{aligned}$$

Terms with the wrong [34] symmetry cancel:

$$(1 - \pi_{34}) (X_{4010} + X_{2210} + X_{2030} + X_{2012}) = 0$$

Remaining part agrees with GS and RNS:

$$K_{\text{1-loop}}^{\text{2B2F}} = X'_{2111} = -\frac{1}{480} \left(s_{13} (u_2 \zeta_3 (\not{k}_2 + \not{k}_3) \zeta_4 u_1) + s_{23} (u_2 \zeta_4 (\not{k}_1 + \not{k}_3) \zeta_3 u_1) \right)$$

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Two-loop massless four-point amplitude

$$\mathcal{A} = \int d^2\Omega_{11} d^2\Omega_{12} d^2\Omega_{22} \prod_{i=1}^4 \int d^2z_i \frac{\exp\left(-\sum_{i,j} k_i \cdot k_j G(z_i, z_j)\right)}{(\det \text{Im } \Omega)^5} K(k_i, z_i)$$

Fermionic zero mode integrals:

$$\begin{aligned} K &= \Delta_{12}\Delta_{34} \langle (\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W_1) \mathcal{F}_{2,mn} \mathcal{F}_{3,pq} \mathcal{F}_{4,rs} \rangle + (\text{perm}(1234)) \\ &\equiv \Delta_{12}\Delta_{34} K_{12} + \Delta_{13}\Delta_{24} K_{13} + \Delta_{14}\Delta_{23} K_{14} \end{aligned}$$

Kinematic factors K_{ij} :

$$K_{12} = 4\langle W_{[1} \mathcal{F}_{2]} \mathcal{F}_{[3} \mathcal{F}_{4]} \rangle + 4\langle W_{[3} \mathcal{F}_{4]} \mathcal{F}_{[1} \mathcal{F}_{2]} \rangle$$

Biholomorphic one-form Δ ,

$$\Delta_{ij} = \Delta(z_i, z_j) = \omega_1(z_i)\omega_2(z_j) - \omega_2(z_i)\omega_1(z_j)$$

Four bosons

(Review of [Berkovits & Mafra (2005)])

All three kinematic factors equivalent; consider

$$\begin{aligned} K_{12}^{4B} &= 4 \langle W_{[1}\mathcal{F}_{2]}\mathcal{F}_{[3}\mathcal{F}_{4]} \rangle|_{4B} + 4 \langle W_{[3}\mathcal{F}_{4]}\mathcal{F}_{[1}\mathcal{F}_{2]} \rangle|_{4B} \\ &= (1 - \pi_{12})(1 - \pi_{34})(1 + \pi_{13}\pi_{24}) \langle W_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4 \rangle|_{4B} \end{aligned}$$

Fermionic expansion:

$$\begin{aligned} \langle W_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4 \rangle|_{4B} &= Y_{5000} + Y_{1400} + Y_{1040} + Y_{1004} + \dots + Y_{1022} \\ &= (1 + \pi_{23})(1 - \pi_{24}) \left(\frac{1}{3}Y_{5000} + Y_{1400} + Y_{3200} + Y_{1022} \right) \end{aligned}$$

Introduce symmetrisation operator:

$$\begin{aligned} K_{12}^{4B} &= \mathcal{S}_{4B} \left(\frac{1}{3}Y_{5000} + Y_{1400} + Y_{3200} + Y_{1022} \right), \\ \mathcal{S}_{4B} &= (1 - \pi_{12})(1 - \pi_{34})(1 + \pi_{13}\pi_{24})(1 + \pi_{23})(1 - \pi_{24}). \end{aligned}$$

Non-zero correlators:

$$\begin{aligned} Y_{3200} &= -\frac{1}{192} k_a^1 F_{cd}^1 k_m^2 F_{ef}^2 F_{pq}^3 F_{rs}^4 \langle (\lambda \gamma^{mnpqr} \lambda)(\lambda \gamma^s \gamma^{ab} \theta)(\theta \gamma_b^{cd} \theta)(\theta \gamma_n^{ef} \theta) \rangle, \\ Y_{1022} &= -\frac{1}{64} F_{ab}^1 F_{mn}^2 k_p^3 F_{cd}^3 k_r^4 F_{ef}^4 \langle (\lambda \gamma^{mnpq[r} \lambda)(\lambda \gamma^{s]} \gamma^{ab} \theta)(\theta \gamma_q^{cd} \theta)(\theta \gamma_s^{ef} \theta) \rangle. \end{aligned}$$

Four bosons (2)

Symmetriser \mathcal{S}_{4B} projects out $k \cdot F$ terms:

$$\mathcal{S}_{4B}(Y_{3200} + Y_{1022}) = -\frac{1}{120}(s_{13} - s_{23})(4 \operatorname{Tr}(F_{(1}F_2F_3F_{4)}) - \operatorname{Tr}(F_{(1}F_2) \operatorname{Tr}(F_3F_{4}))$$

Very simple result:

$$K_{12}^{4B} = -\frac{1}{720}(s_{13} - s_{23})t_8 F^4$$

Trivially obtain K_{13} and K_{14} . Total:

$$K_{2\text{-loop}}^{4B} = -\frac{1}{720}((s_{13} - s_{23})\Delta_{12}\Delta_{34} + (s_{12} - s_{23})\Delta_{13}\Delta_{24} + (s_{12} - s_{13})\Delta_{14}\Delta_{23})t_8 F^4$$

Agrees with (relatively recent!) RNS result [d'Hoker, Phong (2001)]. Note product structure, with one half being the one-loop kinematic factor

Four fermions

Very similar calculation:

$$\begin{aligned} K_{12}^{4F} &= (1 - \pi_{12})(1 - \pi_{34})(1 + \pi_{13}\pi_{24}) \langle W_1 F_2 F_3 F_4 \rangle|_{4F} \\ &= 4(1 - \pi_{12}) \langle W_1 F_2 F_3 F_4 \rangle|_{4F} = \mathcal{S}_{4F} \left(\frac{1}{3} Y_{2111} + Y_{0311} \right) \end{aligned}$$

with symmetrisation $\mathcal{S}_{4F} = 4(1 - \pi_{12})(1 - \pi_{23})(1 + \pi_{24})$.

Two correlators have to be computed:

$$\begin{aligned} Y_{2111} &= (-2) k_a^1 k_m^2 k_p^3 k_r^4 \\ &\quad \times \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} \gamma^{ab} \theta) (\theta \gamma_b u_1) (\theta \gamma_n u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle \\ &= \frac{1}{5040} (-19, -21, 21, 19, -17, -17, 0, 0, 0, 0)_{B_1 \dots B_{10}}, \\ Y_{0311} &= \left(-\frac{2}{3}\right) k_a^2 k_m^2 k_p^3 k_r^4 \\ &\quad \times \langle (\lambda \gamma^{mnpq[r} \lambda) (\lambda \gamma^{s]} u_1) (\theta \gamma_n^{ab} \theta) (\theta \gamma_b u_2) (\theta \gamma_q u_3) (\theta \gamma_s u_4) \rangle \\ &= \frac{1}{15120} (-14, -16, 0, 2, -8, -8, 0, -5, -5, 0)_{B_1 \dots B_{10}}. \end{aligned}$$

Four fermions (2)

Act with symmetriser S_{4F} and obtain the one-loop μ^4 scalar:

$$\begin{aligned} K_{12}^{4F} &= S_{4F}\left(\frac{1}{3}Y_{2111} + Y_{0311}\right) = \frac{1}{45}(-1, 0, 1, 0, -1, 0, 0, 0, 0, 0)_{B_1 \dots B_{10}} \\ &= \frac{1}{45} s_{12} ((u_1 k_3 u_2)(u_3 k_1 u_4) - (u_1 k_2 u_3)(u_2 k_1 u_4) + (u_1 k_2 u_4)(u_2 k_1 u_3)) . \end{aligned}$$

Total result, including K_{13} and K_{14} :

$$\begin{aligned} K_{\text{2-loop}}^{4F} &= \frac{1}{45} \cdot (s_{12} \Delta_{12} \Delta_{34} - s_{13} \Delta_{13} \Delta_{24} + s_{14} \Delta_{14} \Delta_{23}) \\ &\quad \times ((u_1 k_3 u_2)(u_3 k_1 u_4) - (u_1 k_2 u_3)(u_2 k_1 u_4) + (u_1 k_2 u_4)(u_2 k_1 u_3)) \end{aligned}$$

Again simple product structure, with one-loop factor

Two bosons, two fermions

Exchange symmetry prevents simple product structure with one-loop factor:

$$K_{\text{2-loop}}^{\text{2B2F}} \neq \sum_{j=1,2,3} \Delta_{1j} \Delta_{kl} \times [f_j(s_{12}, s_{13}) K_{\text{1-loop}}^{\text{2B2F}}(f_1 f_2 b_3 b_4)].$$

(Note that any $f(s_{12}, s_{13})$ has identical symmetry under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, and is symmetric under $(1, 3) \leftrightarrow (2, 4)$).

Instead, find

$$\begin{aligned} K_{12}^{\text{2B2F}} &= (\mathbf{1} - \pi_{12})(\mathbf{1} - \pi_{34})\tilde{K} \\ K_{13}^{\text{2B2F}} &= (2 \cdot \mathbf{1} + \pi_{12} + \pi_{34} + 2\pi_{12}\pi_{34})\tilde{K} \\ K_{14}^{\text{2B2F}} &= (\mathbf{1} + 2\pi_{12} + 2\pi_{34} + \pi_{12}\pi_{34})\tilde{K} \end{aligned}$$

with

$$\tilde{K} = \langle (\lambda^3 W_1^{(\text{even})}) \mathcal{F}_2^{(\text{odd})} \mathcal{F}_3^{(\text{even})} \mathcal{F}_4^{(\text{even})} \rangle + \langle (\lambda^3 W_3^{(\text{odd})}) \mathcal{F}_4^{(\text{even})} \mathcal{F}_1^{(\text{odd})} \mathcal{F}_2^{(\text{odd})} \rangle$$

Two bosons, two fermions (2)

Correlators in \tilde{K} : combination of ten $k^3 u_1 u_2 F_3 F_4$ scalars.
 Images of the symmetrisers are spanned by s_{12}, s_{13} times

$$(u_1 \gamma^a u_2) (k_a^3 F_{bc}^3 F_{bc}^4 - 2F_{ab}^3 F_{bc}^4 k_c^1 + 2F_{ab}^4 F_{bc}^3 k_c^1)$$

Tensor structure thus fixed up to three constants:

$$\begin{aligned} K_{\text{2-loop}}^{\text{2B2F}} &= [c_1 s_{12} \Delta_{12} \Delta_{34} + (c_2 s_{12} + c_3 s_{13}) \Delta_{13} \Delta_{24} \\ &\quad + ((c_3 - c_2) s_{12} + c_3 s_{13}) \Delta_{14} \Delta_{23}] \\ &\quad \times (u_1 \gamma^a u_2) (k_a^3 F_{bc}^3 F_{bc}^4 - 2F_{ab}^3 F_{bc}^4 k_c^1 + 2F_{ab}^4 F_{bc}^3 k_c^1) \end{aligned}$$

Discussion

Summary:

- The pure spinor formalism yields manifestly supersymmetric loop amplitudes
- Superspace integration not (much) more complicated for fermions than for bosons
- Some amplitudes fixed or highly constrained by exchange symmetries

Outlook:

- Improve computer algebra: custom-made Mathematica algorithms vs. specialised program (e.g. Cadabra)
- Methods applicable to future higher-loop amplitudes