Stability and duality in $\mathcal{N} = 2$ supergravity

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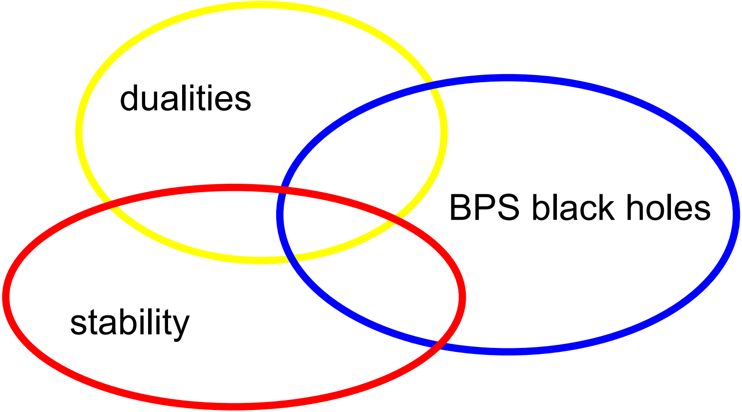
Introduction



Outline

- BPS-states and wall-crossing
- Review supergravity and dualities
- (Multi)-center black hole solutions
- Partition functions
- Convergence and modularity
- Conclusions

Based on arXiv:0906.1767 and 1003.1570

BPS-states and wall-crossing I

$\mathcal{N} = 2$ algebra

$\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} = 2\varepsilon_{\alpha\beta} Z^{IJ}$,

central charge: $Z : (L, C_X) \to \mathbb{C}$, where:

- L: lattice of electro-magnetic charges
- C_X: moduli space

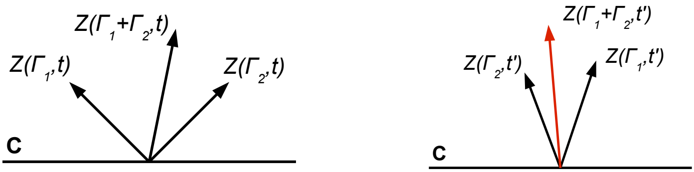
BPS states:

- invariant under half of the susy generators,
- their mass satisfies $M = |Z(\Gamma, t)|$ with $\Gamma \in L$ and $t \in C_X$

Supersymmetric index:

$$\Omega(\Gamma; t) = \frac{1}{2} \mathrm{Tr}_{\mathcal{H}(\Gamma, t)} \, (2J_3)^2 \, (-1)^{2J_3}$$

Is generically a protected quantity.



$\Omega(\Gamma_1 + \Gamma_2; t)$ is only locally constant as function of t; it might jump across walls where $Z(\Gamma_1, t) || Z(\Gamma_2, t)$.

Wall-crossing formulas:

Primitive:

$$\Delta\Omega(\Gamma_1+\Gamma_2; t_{\rm s}\to t_{\rm u}) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle}\,|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(\Gamma_1;t_{\rm ms})\,\Omega(\Gamma_2;t_{\rm ms}),$$

Denef, Moore (2007)

Kontsevich-Soibelman formula:

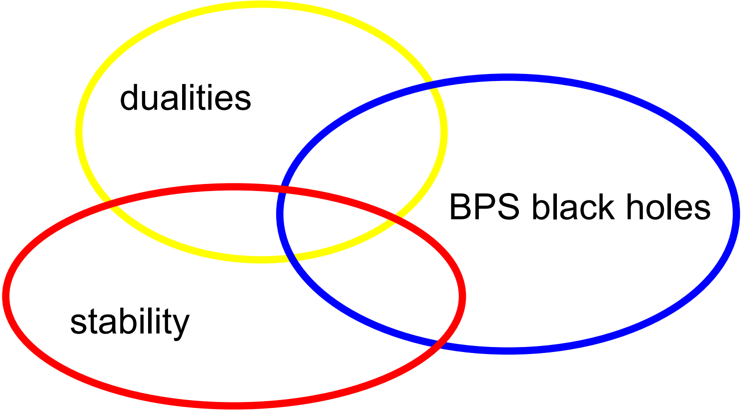
$$\prod^{\curvearrowright}_{\Gamma\in L,\, Z(\Gamma,t)\in V} T_\Gamma^{\Omega(\Gamma;t)}$$

Partition function:

Mixed ensemble:

$$\mathcal{Z}(\tau, C, t) \;=\; \sum_{Q} \Omega(P, Q; t)\, e^{-2\pi\tau_2 M(\Gamma,t) + 2\pi i C^A Q_A}$$

$\tau_2 \in \mathbb{R}_+,\ C^A \in \mathbb{R}^{b_2+1}$


dualities
BPS black holes
stability

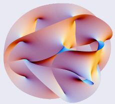
Relevant field content:

vector multiplets:

- $U(1)$ field strengths $F^A = dC^A$, $A = 0, \ldots, b_2$ sourced by electro-magnetic charges $\Gamma = (P, Q) \in L$,
- complex scalars X^A
- fermions

Compactification

compactify 10d space-time on a Calabi-Yau 3-fold X (6 real dimensions) $\Longrightarrow \mathcal{N} = 2$ supergravity in $\mathbb{R}^{1,3}$



$\mathcal{N} = 2$ supergravity II

Properties of X:

- Betti numbers $b_n = \dim H_n(X)$: $b_0 = b_6 = 1$, $b_2 = b_4$, b_3, $b_1 = b_5 = 0$
- triple intersection product of 4-cycles: d_{abc}
- Kähler moduli: $t^a = X^a/X^0 = B^a + iJ^a$, $a = 1, \ldots, b_2$
- Kähler cone:
  $C_X = \{J : Q \cdot J > 0, P \cdot J^2, J^3 > 0 \text{ for } Q, P \text{ effective}\}$

Charges

$\Gamma = (P^0, P^a, Q_a, Q_0) =$ D6-D4-D2-D0 branes $\in H_6 \oplus H_4 \oplus H_2 \oplus H_0$

Symplectic inner product:
$\langle \Gamma_1, \Gamma_2 \rangle = I_{12} = -P_1^0 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q_1 + P_2^0 Q_{0,1}$

Electric-magnetic duality:

- Electric-magnetic duality is a symplectic group $Sp(2b_2+2,\mathbb{Z})$: $\mathbf{K}^{\mathrm{T}}\mathbf{I}\mathbf{K}=\mathbf{I}$. Acts on the vector multiplets, e.g. $\mathbf{K}\Gamma$
- Large volume limit $J\to\infty$: subgroup of translations $\mathbf{K}(k)\sim k^a\in\mathbb{Z}^{b_2}$

$$\begin{aligned}
Q_a &\to Q_a + d_{abc}k^bP^c\\
Q_0 &\to Q_0 + k\cdot Q + \frac{1}{2}d_{abc}k^ak^bP^c\\
t^a &\to t^a + k^a
\end{aligned}$$

- Large gauge transformations $C^a\to C^a+m^a$, also $\mathbb{Z}^{b_2}$

Dualities II

- $SL(2,\mathbb{Z})$ duality group: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$
  More manifest in IIB supergravity $\rightarrow$ T-duality on S^1_{t}
- How does it act?

  $\tau = \tau_1 + i\tau_2 = C_0 + i\beta/g_{\mathrm{s}}$

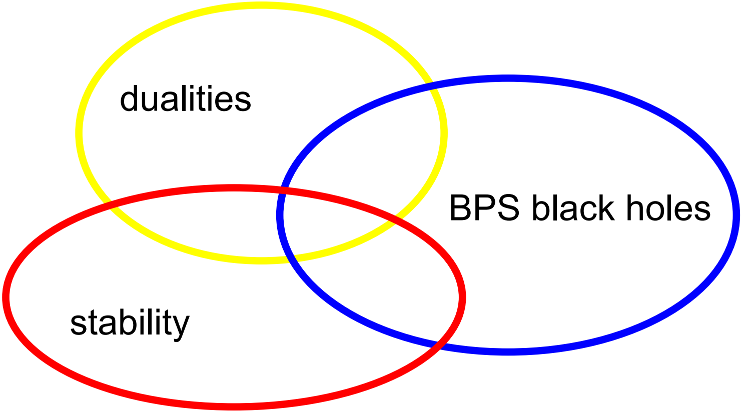
  $\tau \rightarrow \frac{a\tau+b}{c\tau+d}, \quad C \rightarrow aC + bB, \quad B \rightarrow cC + dB, \quad J \rightarrow |c\tau + d| J$
- S-duality + EM-duality + large gauge transformations $\rightarrow$
  Jacobi group $SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^{b_2})^2$

Dualities III

Dualities $\Rightarrow$ modular properties of partition function:

$$\mathcal{Z}(\gamma(\tau, C, t)) \sim \mathcal{Z}(\tau, C, t), \quad \gamma \in SL(2,\mathbb{Z})$$
$$\mathcal{Z}(\tau, C+k, t+\ell) \sim \mathcal{Z}(\tau, C, t)$$

$\Rightarrow$ Partition function is useful to test the compatibility of stability with duality.


dualities
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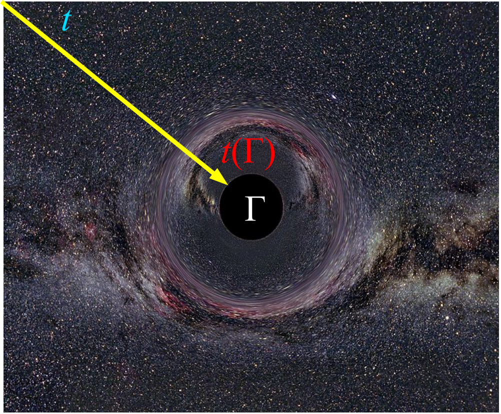
Single center BPS black hole

Moduli "flow" by attractor mechanism. Ferrara, Kallosh and Strominger (1995)



Mass: $M = |Z(\Gamma, t)|$ Entropy: $S_{\mathrm{BH}}(\Gamma) = \pi |Z(\Gamma, t(\Gamma))|^2$

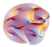
Two center black hole

$$|x_1 - x_2| = \sqrt{G_4}\frac{\langle\Gamma_1,\Gamma_2\rangle}{2}\frac{|Z(\Gamma_1+\Gamma_2,t)|}{\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t))}$$

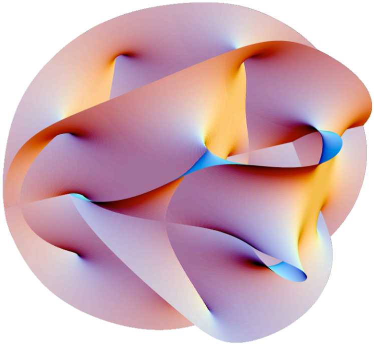
Goal: construction of partition function for such BPS-states.
$\Rightarrow$ test dualities.

Simplification 1

Large volume limit:



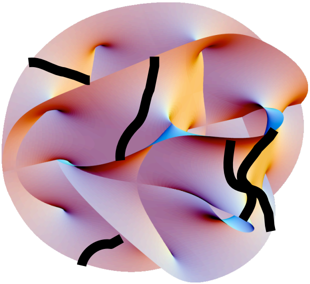
$$\lim J \longrightarrow \infty$$



D-branes $\rightarrow$ coherent sheaves.

Simplification 2

No D6-branes: $P^0 = 0$



D4-brane wraps divisor in X.

$$\Rightarrow Z(\Gamma, t) = -\frac{1}{2} P \cdot t^2 + Q \cdot t - Q_0$$

Single center

Lattice Λ for $Q_a \in H_2(X, \mathbb{Z})$

- quadratic form $D_{ab} = D_{abc}P^c : H_2(X, \mathbb{Z}) \otimes H_2(X, \mathbb{Z}) \to \mathbb{Z}$,
- signature $(1, b_2 - 1)$
- projection to Λ_+: $Q_+ = Q \cdot J/|J|$; such that $Q^2 = Q_-^2 + Q_+^2$

Entropy from MSW CFT Maldacena, Strominger, Witten (1997) :

Entropy: $S_{\rm BH} = \pi\sqrt{\frac{2}{3}P^3\hat{Q}_{\bar{0}}}$

Lower bound: $\hat{Q}_{\bar{0}} = -Q_0 + \frac{1}{2}Q^2 \geq -c_{\rm R}/24 \approx -P^3/24$

To every black hole center a lattice Λ_i, index $\Omega(\Gamma_i) = \Omega(\Gamma_i, t(\Gamma_i))$ and central charge $c_{{\rm R}i}$ is associated.

Mass for single center

Mass

$$\lim_{J\to\infty} M(\Gamma,t) = \frac{1}{2}P\cdot J^2 + Q_+^2 - Q_0$$
$$= \frac{1}{2}P\cdot J^2 + Q_+^2 - \frac{1}{2}Q^2 + \hat{Q}_{\bar{0}}$$

$$\begin{vmatrix} Q^2 & Q\cdot J \\ Q\cdot J & J^2 \end{vmatrix} < 0 \qquad \text{implies} \qquad Q_+^2 - \tfrac{1}{2}Q^2 > 0,$$

$\Rightarrow$ $M(\Gamma,t)$ bounded from below.

Mass for 2 centers

$$\lim_{J\to\infty} M(\Gamma, t) = \tfrac{1}{2} P \cdot J^2 + Q_+^2 - Q_0$$
$$= \tfrac{1}{2} P \cdot J^2 + \underbrace{Q_+^2 - \tfrac{1}{2}(Q_1)_1^2 - \tfrac{1}{2}(Q_2)_2^2}_{\text{signature}\,(2b_2-1,1)} + \hat{Q}_{\bar{0},1} + \hat{Q}_{\bar{0},2}$$

$\Rightarrow$ not bounded from below for generic $Q_1 \in \Lambda_1$, $Q_2 \in \Lambda_2$.

Stability

$(P_1 \cdot Q_2 - P_2 \cdot Q_1)\, \mathrm{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t)) < 0$

$\Rightarrow Q_+^2 - \tfrac{1}{2}(Q_1)_1^2 - \tfrac{1}{2}(Q_2)_2^2 > 0$

$\Rightarrow$ bounded from below.

Partition function for single center I

For single center:

Mass bounded from below
$\Rightarrow \mathcal{Z}_P(\tau, C, t) = \sum_{Q_I} \Omega(\Gamma)\, e^{-\pi\tau_2 M(\Gamma,t) + 2\pi i C^I Q_I}$ is convergent.

S-duality/modularity

$$\begin{aligned} S &: \quad \mathcal{Z}_P(-1/\tau, -B, C + i|\tau| J) = \tau^{\frac{1}{2}} \bar{\tau}^{-\frac{3}{2}}\, \varepsilon(S)\, \mathcal{Z}_P(\tau, C, t), \\ T &: \quad \mathcal{Z}_P(\tau + 1, C + B, t) = \varepsilon(T)\, \mathcal{Z}_P(\tau, C, t), \end{aligned}$$

Partition function for single center II

Electric-magnetic duality

$$\mathcal{Z}_P(\tau, C, t+k) = (-1)^{P\cdot k}\, e(C\cdot k/2)\, \mathcal{Z}_P(\tau, C, t),$$
$$\mathcal{Z}_P(\tau, C+k, t) = (-1)^{P\cdot k}\, e(-B\cdot k/2)\, \mathcal{Z}_P(\tau, C, t).$$

Theta function decomposition:

$\mathcal{Z}_P(\tau, C, t) = \sum_{\mu\in\Lambda^*/\Lambda} \overline{h_{P,\mu}(\tau)}\, \Theta_\mu(\tau, C, B),$

vector-valued modular form:
$h_{P,\mu}(\tau) = \sum_{Q_0} \Omega(\Gamma) q^{-Q_0+\frac{1}{2}Q^2}, \qquad Q \in \mu + \Lambda$

Siegel-Narain theta function:
$\Theta_\mu(\tau, 0, 0) = \sum_{Q\in\Lambda+\mu} (-1)^{P\cdot Q} \exp\left(\pi i(\tau Q_+^2 + \bar{\tau} Q_-^2)\right)$

Partition function for 2 centers I

How to implement:

$$\Omega_{P_1\leftrightarrow P_2}(\Gamma;t) = \tfrac{1}{2}(\mathrm{sgn}(\mathrm{Im}\, Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t)) + \mathrm{sgn}(\langle\Gamma_1,\Gamma_2\rangle)) \times \langle\Gamma_1,\Gamma_2\rangle\; (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\,\Omega(\Gamma_1)\,\Omega(\Gamma_2).$$

Partition function:

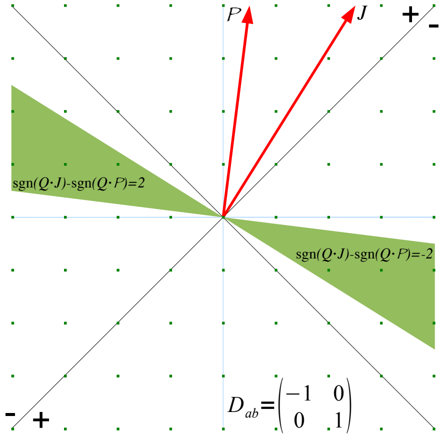
$$\mathcal{Z}_{P_1\leftrightarrow P_2}(\tau,C,t) = \sum_{(\mu_1,\mu_2)\in\Lambda_1^*/\Lambda_1\oplus\Lambda_2^*/\Lambda_2} \overline{h_{P_1,\mu_1}(\tau)}\,\overline{h_{P_2,\mu_2}(\tau)}\,\Psi_{(\mu_1,\mu_2)}(\tau,C,B)$$

$\Psi_{(\mu_1,\mu_2)}(\tau,C,B)$ is a combination of a Siegel-Narain theta function and indefinite theta function

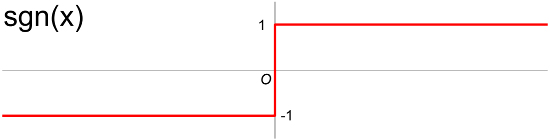
Indefinite theta function I

Sums only over negative definite lattice points Göttsche, Zagier (1996); Zwegers (2002):

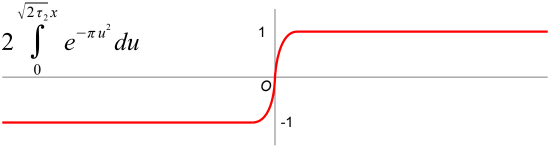
$$\Theta_\mu(\tau) = \sum_{Q\in\mu+\Lambda} \tfrac{1}{2}(\mathrm{sgn}(Q\cdot J) - \mathrm{sgn}(Q\cdot \mathcal{P}))\exp(2\pi i\bar\tau Q^2/2)$$



Modular invariant? No, but mock modular invariant.
$\Theta_\mu(\tau) \to \Theta^*_\mu(\tau)$ by replacing



with



approaches $\mathrm{sgn}(x)$ for $\tau_2 \to \infty$.

Partition function for 2 centers II

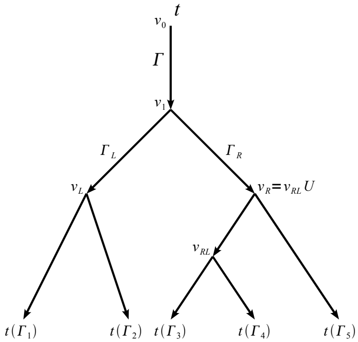
Similarly $\Psi_{(\mu_1,\mu_2)}(\tau, C, B) \to \Psi^*_{(\mu_1,\mu_2)}(\tau, C, B)$

$\Rightarrow$ then $\mathcal{Z}_{P_1 \leftrightarrow P_2}(\tau, C, t)$:

- has same modular properties as $\mathcal{Z}_P(\tau, C, t) \to$ evidence for compatibility of stability and duality
- is continuous across walls, reminiscent of results by Gaiotto, Moore, Neitzke (2008); Joyce (2006)

Flow trees I

Flow trees are schematic representations of supergravity solutions.



Analysis of more complicated BPS objects is possible.

Flow trees II

- Also the contribution of flow trees with 3 endpoints is convergent.
- partition functions with manifest S-duality, are generating functions of
$$\bar{\Omega}(\Gamma; t) = \sum_{m|\Gamma} \frac{\Omega(\Gamma; t)}{m^2}$$
instead of $\Omega(\Gamma; t)$
- $\bar{\Omega}(\Gamma; t)$ seem most natural to determine the contribution of flow trees.

The end

Evidence is given for:

- the convergence of the BPS partition function in the mixed ensemble,
- the compatibility of stability and duality
- this also makes the partition function continuous of t

Open problems:

- modularity for $N \geq 3$
- relax $P^0 = 0$ and $J \to \infty$