BLACK HOLE SCATTERING

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Outline

- Introduction
- Overview of the Technique

  ODEs + Singularities + Monodromies + Boundary conditions

Results

- Kerr BH Scattering: a systematic study
- Schwarzschild BH Scattering: Quasi-normal modes

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To appear

Introduction

Black Holes rule!

Black holes are the most intriguing objects in the universe.

They rule and shape galaxies; the experimental data gathered so far has not only confirmed their existence but also shown their effects on their environment.

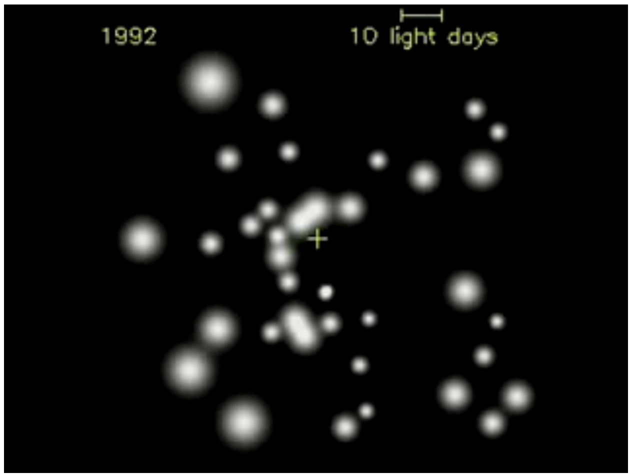
These objects are actually very simple solutions of General Relativity.

And in spite of all the progress in the description of black holes, a full understanding of the physics of black holes has not been yet achieved.

Over the last 40 years there has been a lot of progress …

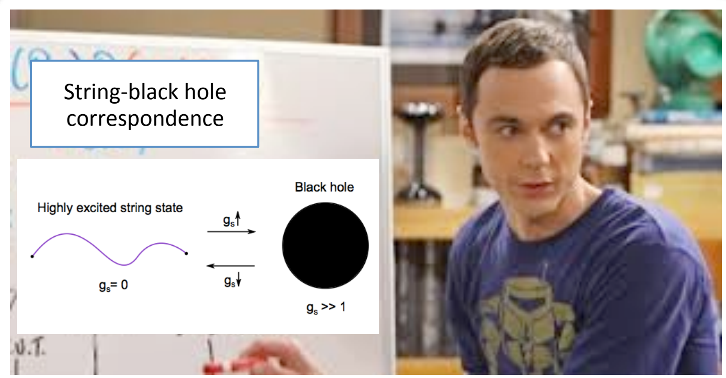
General Relativity

Astrophysics



$$ds^2 = \frac{\Sigma}{\Delta}dr^2 - \frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\, d\phi - a\, dt\right)^2 \ ,$$

$$\Delta = r^2 + a^2 - 2Mr \ , \quad \Sigma = r^2 + a^2\cos^2\theta \ .$$



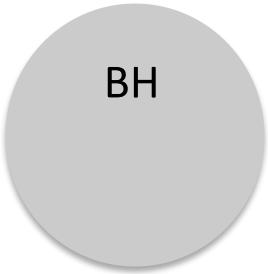
String Theory

Black holes are thermal systems.

They can be characterized by its temperature and entropy.

T_H

S_H



This thermodynamical aspect raises some intriguing questions

1. What is the microscopical interpretation of the black hole entropy?
2. As radiating (thermal)bodies, which is the characteristic emission rate? and, what information from the black hole does it carry?
3. Black Holes can also scatter radiation so one could ask how? In the process what thermal information can be extracted?

All these three questions have been tackled and partially answered in one of another way from a string theoretical perspective with great success.

1. The entropy counting of microstates (in e.g. extremal and near extremal black holes)

2. The greybody factors and the agreement with the D-brane emission rate

3. Due to AdS/CFT, quasi-normal frequencies in the BTZ black hole spacetime yield a prediction for thermalization timescale in the dual two-dimensional CFT, which otherwise would be very difficult to compute directly.

The answers are not complete, but we are making steps towards a better understanding of these problems!

BLACK HOLE SCATTERING

The scattering process of any field by a black hole space-time is characterized by the scattering coefficients

These have been fully understood in special cases – the low frequency regimes.

Q1: Can we do better?

E.g. Can we solve for all values of the frequencies ?

Q2: Do we understand why the problem is so challenging ?

Not really… although we made progress in this direction.

Main purpose: provide new insights into the properties of the scattering coefficients and the link to the global information of the the wave equation.

Astrophysics:

Solutions of the KG equation may become a powerful tool to construct the theoretical templates towards LIGO and VIRGO projects.

Mathematics:

New studies for solving the problem in full generality are required. This may prompt the progress in the connection problem of linear complex differential equations.

Kerr/CFT:

In the limits where the solutions of the KG equation are hypergeometric functions the BH is conjectured to have dual CFT description. A deeper understanding of the solutions in other regimes may be crucial to this problem.

Technique

Our goal is to capture the dependence of scattering coefficients and quasi-normal modes on analytic (global) properties of linearized fluctuations around a black hole background in asymptotically flat space-time.

We will …

… analyze the singularities of the KG. wave equation,

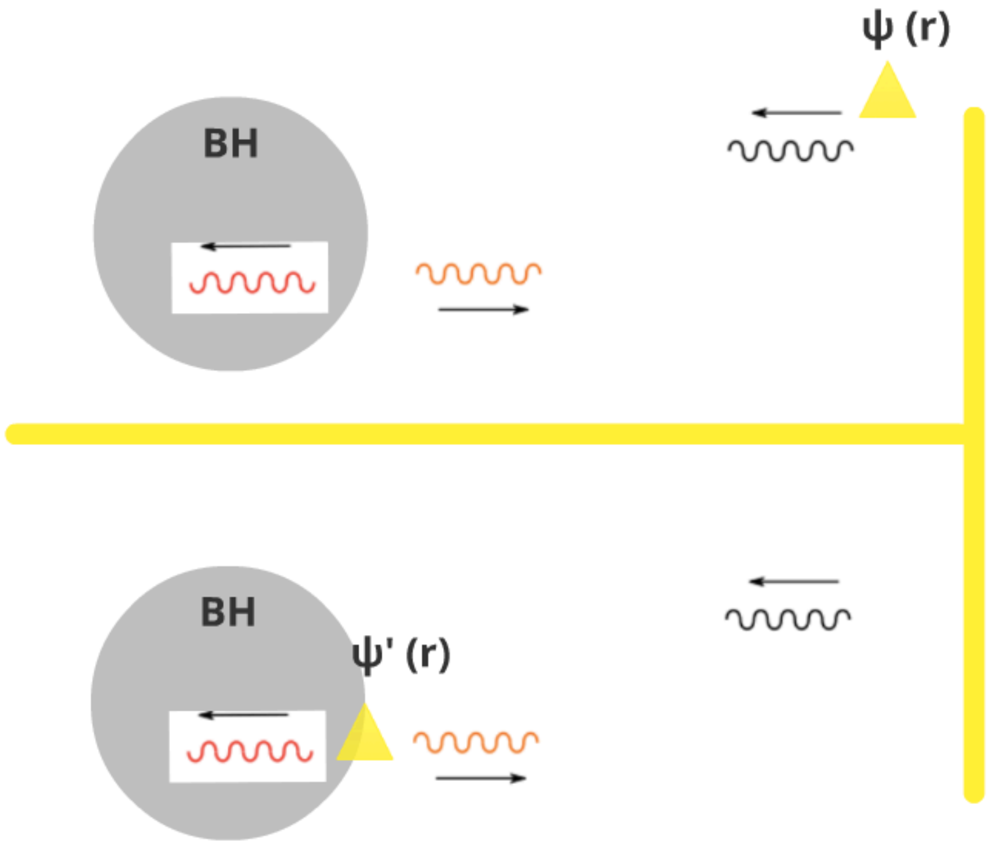
… solve the scattering PDEs problem with particular boundary conditions …

The choice of boundary conditions is intimately connected to the analytic properties of solutions when the independent variable (e.g., the radial direction) is complexified.

… and see how the knowledge of the analytic properties of the solutions can be used to compute scattering coefficients, turning a portion of the hard calculations into simpler exercises in linear algebra.

Black hole Scattering

Recall



$$\begin{pmatrix} \psi'_w(r) \\ \psi'_{-w}(r) \end{pmatrix} = \begin{pmatrix} 1/T & R/T \\ R^*/T^* & 1/T^* \end{pmatrix} \begin{pmatrix} \psi_w(r) \\ \psi_{-w}(r) \end{pmatrix}$$

R: reflexion coef.
T: trans. coef.

We want to find R or T.

The wave equation

Consider the KG equation of a massless scalar in a BH background

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0\ .$$

that is separable so that the eq. reduces to solving a second order ODE.

Note: So far the only solutions found are in the low frequency limit.

Our method provides the tools to perform the computations for all ranges of the frequency.

$$\partial_z\left(W(z)\partial_z\psi(z)\right) - V(z)\,\psi(z) = 0\ . \quad *$$

* This can always be rewritten as two coupled first order ODEs

$$\partial_z\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{W(z)}\\ V(z) & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} =: A(z)\Psi\ ,$$

There ODE

$$\partial_z \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{U(z)} \\ V(z) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} =: A(z)\Psi \ ,$$

where

$$\Psi_1 = \psi$$

$$\Psi_2 = U(z)\partial_z\psi.$$

There is a two-dimensional space of solutions to this ODE, so we can choose a linearly independent basis of solutions, Ψ(1) and Ψ(2), and collect them into a so-called fundamental matrix

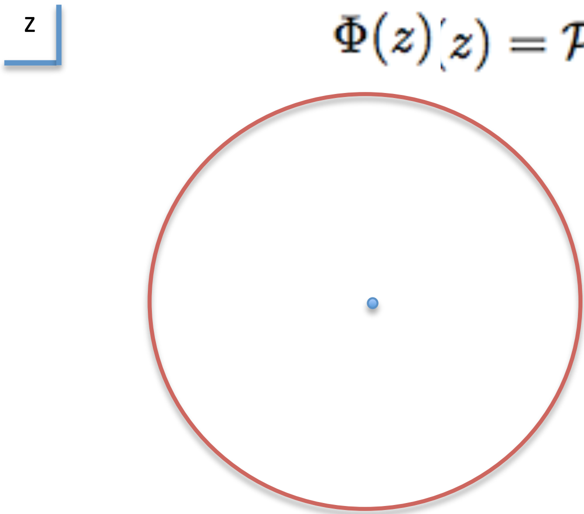
$$\Phi(z) := \begin{pmatrix} \Psi^{(1)} & \Psi^{(2)} \end{pmatrix}$$

where linear independence of the two solutions is equivalent to the invertibility of Φ(z).

We analytically continue to the complex z-plane and restrict to cases where A(z) is meromorphic — all cases considered here certainly satisfy this requirement.

Follow Φ(z) around a closed loop γ in the complex z-plane, calling the result

$$\Phi(z)(z) = \mathcal{P}\{e^{\oint_\gamma A}\}\Phi(z) =: \Phi(z)M_\gamma\,,$$



Since A(z) is meromorphic, the differential operator $\partial_z - A$ returns to itself, which implies that $\Phi_\gamma(z)$ must again be a fundamental matrix for the ODE, however it need not be equal to Φ(z): given one fundamental matrix, we can always multiply it from the right by a constant invertible matrix to obtain another (i.e., we can choose a different linearly independent basis of solutions). By the definition of a fundamental matrix, then,

Monodromy matrix $$M_\gamma = \Phi(z)^{-1}\mathcal{P}\{e^{\oint_\gamma A}\}\Phi(z)$$

In particular, if one can find a gauge in which A has no poles enclosed by γ, then

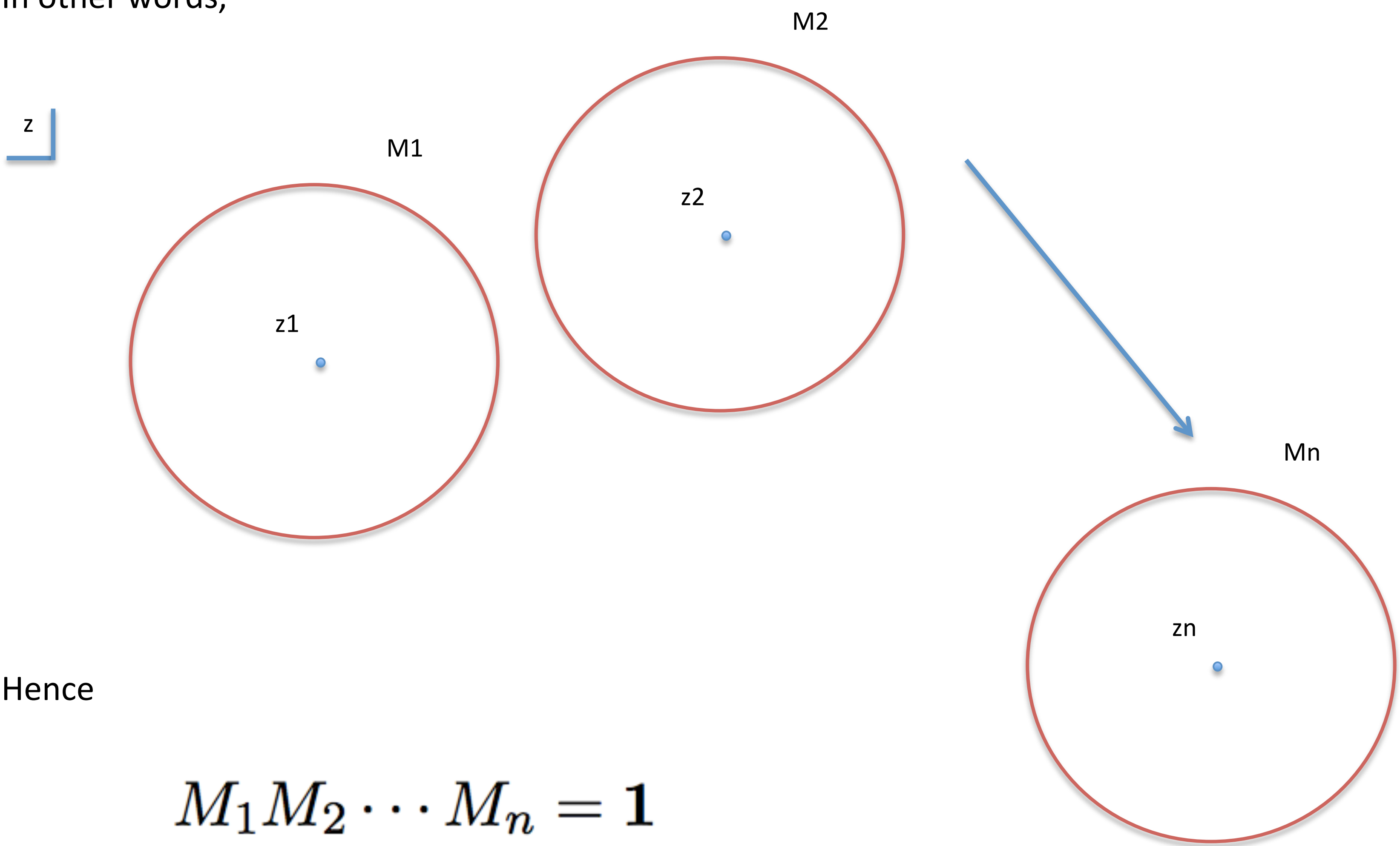
$$M_\gamma = 1.$$

In this way, poles of A that cannot be removed by gauge transformations correspond to branch points of Φ(z) and are associated with non-trivial monodromy matrices.

This has an implication that is key to the rest of our study.

Let $z = z_i$ for $i = 1,\dots,n$, be the locations of all branch points of Φ(z), and let M_i be the monodromy matrix associated with a loop that encloses only the branch point at z_i. If we follow Φ(z) around a path enclosing all branch points, the other side of the loop encloses no branch points and so the monodromy around that loop must be trivial.

In other words,



Hence

$$M_1 M_2 \cdots M_n = \mathbf{1}$$

The conjugacy class of each individual Mi can often be computed quite simply from local information of the differential equation (with an important caveat which we will see later), while

$$M_1 M_2 \cdots M_n = \mathbf{1}$$

is a relation among these local data — it is a piece of global information

Computing scattering coefficients is an example of a problem where we require global information about our solutions (relating boundary conditions at different points in the z plane), so we explore how we may exploit this relation.

As we alluded to above, if Φ has a branch point at zi, then A has a pole there.

The converse is not true because it may be possible to remove the pole in A by a gauge transformation.

Fortunately, there exists a simple algorithm for choosing gauge transformations to reduce the order of the pole to some minimal integral value called the Poincare rank, $\mathrm{R}_i \in \mathbb{N}_0$, and a gauge where A takes the form

$$A(z) = (z - z_i)^{-\mathrm{R}_i - 1} A_0(z) \, ,$$

has a convergent Taylor series expansion in some neighborhood containing z = zi

Types of poles

A simple pole Ri = 0 is called a regular singular point,

A higher order pole Ri > 0 is called an irregular singular point of rank ri.

The distinction between regular and irregular singular points might seem artificial, but their implications for Φ are starkly different:

regular singular points correspond to algebraic or logarithmic branch points in Φ, while irregular singular points correspond to essential singularities in Φ and exhibit Stokes' phenomenon.

Understanding regular singular points will suffice to illustrate the essence of our approach to scattering computations, so we focus first on this case.

We will then get into the complications of the irregular singular points and Stokes phenomenon.

Monodromies and Regular Singular Points

Our goal now is to find monodromies and the local behavior of Φ around singular points.

To find the conjugacy class of Mi, first perform a gauge transformation to bring A to the minimal form. Once that is done the conjugacy class of the monodromy is easily determined

$$M_i \cong \mathcal{P}\{e^{\oint_{\gamma_i} A}\} \cong e^{\oint_{\gamma_i} A},$$

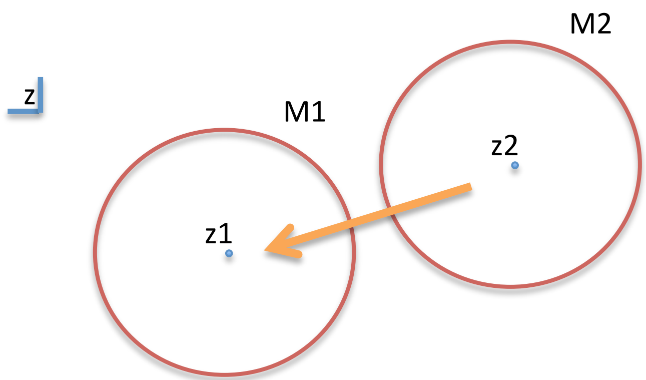
For simplicity, suppose that Mi has distinct eigenvalues e∓2παi (so Ni has eigenvalues ±iαi), then we are free to choose our fundamental matrix to diagonalize Mi, in which case

$$\Phi(z) = \Big(\Phi_0 + O(z - z_i)\Big) \begin{pmatrix} (z - z_i)^{i\alpha_i} & 0 \\ 0 & (z - z_i)^{-i\alpha_i} \end{pmatrix}$$

Approaching z_i from a direction where $\mathrm{Im}[i\alpha_i \ln(z - z_i)] \neq 0$, we see explicitly that one column corresponds to ingoing waves and the other to outgoing waves.

Diagonalizing the monodromy matrix M_i therefore corresponds to choosing a basis with definite boundary conditions at z_i.

A scattering computation typically involves finding the change of basis between solutions that are ingoing/outgoing at one singular point and solutions that are ingoing/outgoing at another singular point.



Since there are only two linearly independent solutions to the ODE, the following matrix must be a constant

$$\mathcal{M}_{2\to1} = \Phi_2^{-1}\Phi_1$$

is nothing more than a change of basis from left-eigenvectors of M2 to left-eigenvectors of M1.

So far we have said nothing of the normalization of the various bases of solutions, so we are free to rescale the columns of Φ1 and Φ2, meaning the change of basis M2→1 is defined only up to multiplication by a diagonal matrix on either side

$$\mathcal{M}_{2\to1} \sim \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \mathcal{M}_{2\to1} \begin{pmatrix} d_3 & \\ & d_4 \end{pmatrix} .$$

forcing M2→1 ∈ SU(1,1) the ambiguity (up to phases) vanishes

$$\mathcal{M}_{2\to1} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\mathcal{R}^*}{\mathcal{T}^*} & \frac{1}{\mathcal{T}^*} \end{pmatrix} , \qquad |\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$$

where R and T are the reflection and transmission coefficients, respectively.

We have now seen how monodromy matrices relate to boundary conditions at regular singular points and we have seen how to compute the conjugacy class of the corresponding monodromy matrices.

We will only study problems with two or three singularities. In these cases the monodromy matrices can be computed.

The more interesting situation, relevant to our discussions of black holes, is when there are three singular points. In this case, knowledge of the conjugacy class of M1, M2, and M3, together with the global relation in M1 M2 M3=1, is enough to reconstruct the matrices themselves in a common basis.

Explicitly,

$$\det(M_i) = 1\,, \qquad \mathrm{tr}(M_i) = 2\cosh(2\pi\alpha_i)\,, \qquad M_i \neq \mathbf{1}\,, \qquad \text{for} \quad i = 1,2,3\,,$$

$$M_1 M_2 M_3 \;=\; \mathbf{1}\;,$$

a common basis is

$$M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 2\cosh(2\pi\alpha_1) \end{pmatrix}\,, \qquad M_2 = \begin{pmatrix} 2\cosh(2\pi\alpha_2) & e^{2\pi\alpha_3} \\ -e^{-2\pi\alpha_3} & 0 \end{pmatrix}\,,$$

$$M_3 = \begin{pmatrix} e^{2\pi\alpha_3} & 0 \\ 2(e^{-2\pi\alpha_3}\cosh(2\pi\alpha_1) - \cosh(2\pi\alpha_2)) & e^{-2\pi\alpha_3} \end{pmatrix}\,.$$

Whether this can be made an SU(1,1) matrix depends on details of the αi, however when it is possible — for instance, when the αi are all real and M2→1 is invertible — we can read off the norm of the transmission coefficient without even computing the required diagonal transformation

$$1 - |\mathcal{R}|^2 = |\mathcal{T}|^2 = \frac{\sinh(2\pi\alpha_1)\,\sinh(2\pi\alpha_2)}{\sinh\pi(\alpha_3+\alpha_1-\alpha_2)\,\sinh\pi(\alpha_3-\alpha_1+\alpha_2)}\,.$$

Of course, ODEs with three regular singular points have hypergeometric functions as solutions and are therefore well understood, and we have verified that these formulas are correct.

The challenge for applying the same methods to scattering off black holes is related to the irregular singular point, which black hole backgrounds have at asymptotic infinity (a consequence of plane waves having essential singularities at infinity)

The basic idea for computing the scattering coefficients will be the same, but there will be additional steps and subtleties.

As we all know, consciously or not, plane waves become essential singularities in the complex plane. It is crucial, then, to study irregular singular points for applications to asymptotically flat black holes.

Monodromies and Irregular Singular Points

Let A(z) have a rank r ≥ 1 singularity at z = ∞, which means there is a gauge in which

$$A = z^{\mathrm{R}-1} A_0(z)\,,$$

where $A_0(z)$ has a convergent Taylor series expansion around z = ∞ in non-negative powers of 1. Furthermore, let A0(∞) have maximal rank, and choose the gauge so that $A_0(\infty)$ is diagonal.

Then there exists a formal fundamental matrix of the form

$$\Phi_f(z) = P(z)e^{\Lambda(z)}\ ,$$

where P(z) is a non-negative power series in 1/z (generally not convergent) and

It would seem that we can read off the monodromy from these solutions to be $e2\pi i\Lambda_0$ (around infinity, $z \to e{-}2\pi iz$ is the positive direction), as before

However, this is not so because the calculation is complicated by the fact that $P(z)$ is just an asymptotic series, not a convergent series.

For this reason, $e2\pi i\Lambda_0$ is called the formal monodromy, but we will shortly see its relation to the monodromy of a true solution.

For describing boundary conditions at $z = \infty$, though, we care about diagonalizing $\Lambda(z)$ (as opposed to the true monodromy) since this describes the dominant behavior of the solutions at $z = \infty$, telling us whether the solutions are ingoing or outgoing.

On the other hand, the quantity entering the product relation for the transmission/reflection coefficients is the true monodromy, so it is crucial to understand the relationship between the two.

The distinction and relation between them is intertwined with the fact that solutions of ODEs around irregular singular points exhibit Stokes' phenomenon.

Stokes' phenomenon.

The defining feature of Stokes' phenomenon is that it arises when one attempts to describe one function (e.g., a fundamental matrix of true solutions) in terms of a function with a different branching structure (e.g., a formal fundamental matrix).

Consider the following formal expression:

$$\Phi_f(z)^{-1}\Phi_f(z) \;=\; e^{-\Lambda(z)}P(z)^{-1}P(z)e^{\Lambda(z)} \,.$$

The important observation to make is that we can only say that

$$P(z)^{-1}P(z) \;\sim\; \mathbf{1} \qquad (\text{as } z \to \infty)\,,$$

This means it could differ from 1 by something with no series expansion around z = ∞, e.g., ez, and this is precisely what happens.

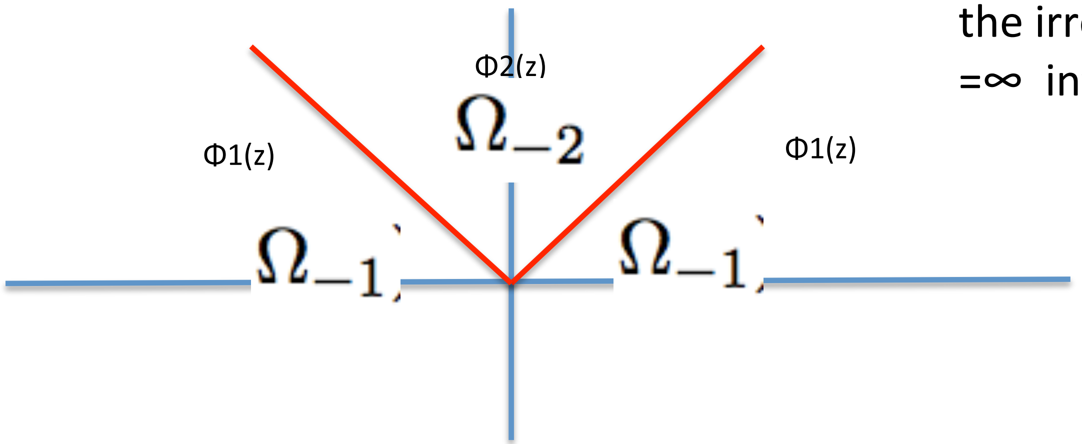
If a product were between two actual fundamental matrices rather than two formal ones, we would expect the result to be a constant matrix.

Again, this arises here because the formal solutions are not convergent series.

Since formal fundamental matrices are asymptotic to actual ones, the limit of $\Phi_f(z)^{-1}\Phi_f(z)$ as z tends to ∞ will be a constant matrix S, called a Stokes matrix, with

$$S_{ij} = \lim_{z\to\infty} e^{-\Lambda_{ii}(z)+\Lambda_{jj}(z)}\left(\delta_{ij} + \underbrace{O(z^{-1})}_{\text{to 0 as z} \to \infty}\right),$$

Schematically



Divide the neighborhood of the irregular singular point at z =∞ into wedges Ω_k

Think of a given asymptotic expansion (e.g., one for which P (∞) = 1) as being asymptotic to a given true solution within a wedge, Ωk. Once we cross the next Stokes ray, the same true solution will have a different formal fundamental matrix

This means that the subdominant solution, the first column of Φ, may be added to the dominant solution, the second column of Φ. Thus,

$$\Phi(z)\big|_{\Omega_{-1}} \sim \Phi_f(z)\big|_{\Omega_{-1}} \left(\begin{smallmatrix} 1 & C_0 \\ 0 & 1 \end{smallmatrix}\right) =: \Phi_f(z)\big|_{\Omega_{-1}} S_0 \qquad (\text{as } z \to \infty,\ z \in \Omega_{-1})\,.$$

At the next overlap, Ω−1 ∩ Ω−2, the roles of dominant and subdominant solutions reverse, so the relevant Stokes matrix will be lower triangular:

$$\Phi(z)\big|_{\Omega_{-2}} \sim \Phi_f(z)\big|_{\Omega_{-2}} \left(\begin{smallmatrix} 1 & 0 \\ C_{-1} & 1 \end{smallmatrix}\right) S_0 =: \Phi_f(z)\big|_{\Omega_{-2}} S_{-1} S_0 \qquad (\text{as } z \to \infty,\ z \in \Omega_{-2})\,,$$

with Ck constant.

This provides us with the identification of the true monodromy as

$$M_\infty = e^{2\pi i \Lambda_0} S_{-2\mathrm{R}+1} \cdots S_0\,.$$

The hard work comes in determining the Ck, commonly called Stokes multipliers.

Their values are not solely determined by the local data of the singularity; Stoke matrices depend on all terms in the connection, including the regular pieces.

Summary

So we learned that to compute the transmission coefficients.

We have to write down the wave equation on a black hole background.

$$\partial_z\big(U(z)\partial_z\psi(z)\big) - V(z)\,\psi(z) = 0 \;\longrightarrow\; \partial_z \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{U(z)} \\ V(z) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} =: A(z)\Psi\ ,$$

Analyze the singularities of A(z)

Compute the monodromies around each singular point. In the cases where there is an irregular singularity – as in any asymotically flat black hole solution– we have to compute the Stokes multipliers too.

Let's see some examples

Results

Kerr BH

Kerr black hole solution

$$ds^2 = \frac{\Sigma}{\Delta}dr^2 - \frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\, d\phi - a\, dt\right)^2$$

$$\Delta = r^2 + a^2 - 2Mr\ , \quad \Sigma = r^2 + a^2\cos^2\theta\ .$$

Wave equation of a massless scalar $\psi(t,r,\theta,\phi) = e^{-i\omega t + im\phi}R(r)S(\theta)$

$$\boxed{\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0\ .}$$

$$\left\{\begin{aligned} &\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - \frac{m^2}{\sin^2\theta} + \omega^2 a^2\cos^2\theta\right]S(\theta) = -K_\ell S(\theta)\ , \\ &\left[\partial_r\Delta\partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r^2 + 2M(r+2M))\omega^2\right]R(r) = K_\ell R(r)\end{aligned}\right.$$

Kerr BH

The “radial” ODE

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r^2 + 2M(r+2M))\omega^2\right] R(r) = K_\ell R(r)$$

where

$$a := M\sqrt{1-\varepsilon^2}\,, \qquad 0 \le \varepsilon \le 1\,,$$
$$r_\pm = M(1 \pm \varepsilon)\,.$$

so ε = 1 is Schwarzschild, ε = 0 is extremal Kerr (where the two horizons merge and become an irregular singular point),

The branch points of solutions of the scalar wave equation for Kerr are located at r = r±,∞. The singularities at the horizons, r±, are regular singular points while the singularity at infinity is an irregular singular point of rank 1

Substitute a series expansion in the ODE

monodromy

$$R(r) = (r - r_+)^{\pm i\alpha_+}\left[1 + O(r - r_+)\right] \longrightarrow \alpha_\pm := \frac{2Mr_\pm\omega - am}{r_+ - r_-}\,.$$

The conjugacy classes of the monodromy matrices associated to the horizons

$$M_+ \cong \begin{pmatrix} e^{-2\pi\alpha_+} & 0 \\ 0 & e^{2\pi\alpha_+} \end{pmatrix}, \qquad M_- \cong \begin{pmatrix} e^{-2\pi\alpha_-} & 0 \\ 0 & e^{2\pi\alpha_-} \end{pmatrix}.$$

Instead, for irregular singular points

$$R(r) = e^{\mp i\omega r} r^{\mp i\lambda - 1}\left[1 + O(r^{-1})\right] ,$$

At the singularity r → ∞ there will be a nontrivial conjugacy class of the monodromy matrix associated to this point as well,

$$M_\infty \cong \begin{pmatrix} e^{-2\pi\alpha_{irr}} & 0 \\ 0 & e^{2\pi\alpha_{irr}} \end{pmatrix} ,$$

But there si the Stokes phenomenon, hence

$$M_\infty = e^{2\pi i \Lambda_0} S_{-1} S_0 \ .$$

Stokes matrices

$$S_{-1} = \begin{pmatrix} 1 & 0 \\ C_{-1} & 1 \end{pmatrix} , \qquad S_0 = \begin{pmatrix} 1 & C_0 \\ 0 & 1 \end{pmatrix} ,$$

Monodromy at the irregular singular point

$$\boxed{\alpha_{irr} = \frac{1}{2\pi} \cosh^{-1}\left[\cosh(2\pi\lambda) + e^{2\pi\lambda} C_0 C_{-1}/2\right] .}$$

The formal monodromy can be read off directly

$$e^{2\pi i \Lambda_0} \cong \begin{pmatrix} e^{-2\pi\lambda} & 0 \\ 0 & e^{2\pi\lambda} \end{pmatrix} , \qquad \lambda = 2M\omega \ .$$

Computing αirr is then equivalent to determining the product C0C−1 which is a quite involved task. Analytically, it is not obvious how to estimate αirr for all frequencies.

We developed the StokesNotebook (for Mathematica) that implements a method to compute the Stokes multipliers, numerically.

Kerr BH

Approximate analytical computation: a perturbative approach for <<1

We can turn this into a first order ODE with connection

$$\hat{A}(z) = \begin{pmatrix} 0 & \frac{1}{z(1-z)} \\ -\frac{\alpha_+^2}{1-z} + \frac{1+4\alpha_-^2}{4} + \frac{1+4K_\ell-(2M\omega)^2(7-4\varepsilon+\varepsilon^2)}{4z} & 0 \end{pmatrix}$$
$$+(2M\omega)^2 \begin{pmatrix} 0 & 0 \\ -\frac{\varepsilon^2}{z^3} - \frac{\varepsilon(2-\varepsilon)}{z^2} & 0 \end{pmatrix}.$$

And find he monodromy at the irregular singular point as a eprturbative expansion

$$\alpha_{irr}^2 = \ell^2 - 4\ell M^2\omega^2 \frac{15\ell(\ell+1)-11}{(2\ell+3)(2\ell+1)(2\ell-1)}$$
$$+16mM^3\omega^3\sqrt{1-\varepsilon^2}\,\frac{5\ell(\ell+1)-3}{(\ell+1)(2\ell+3)(2\ell+1)(2\ell-1)} + O(\omega^4)\,.$$

Kerr BH

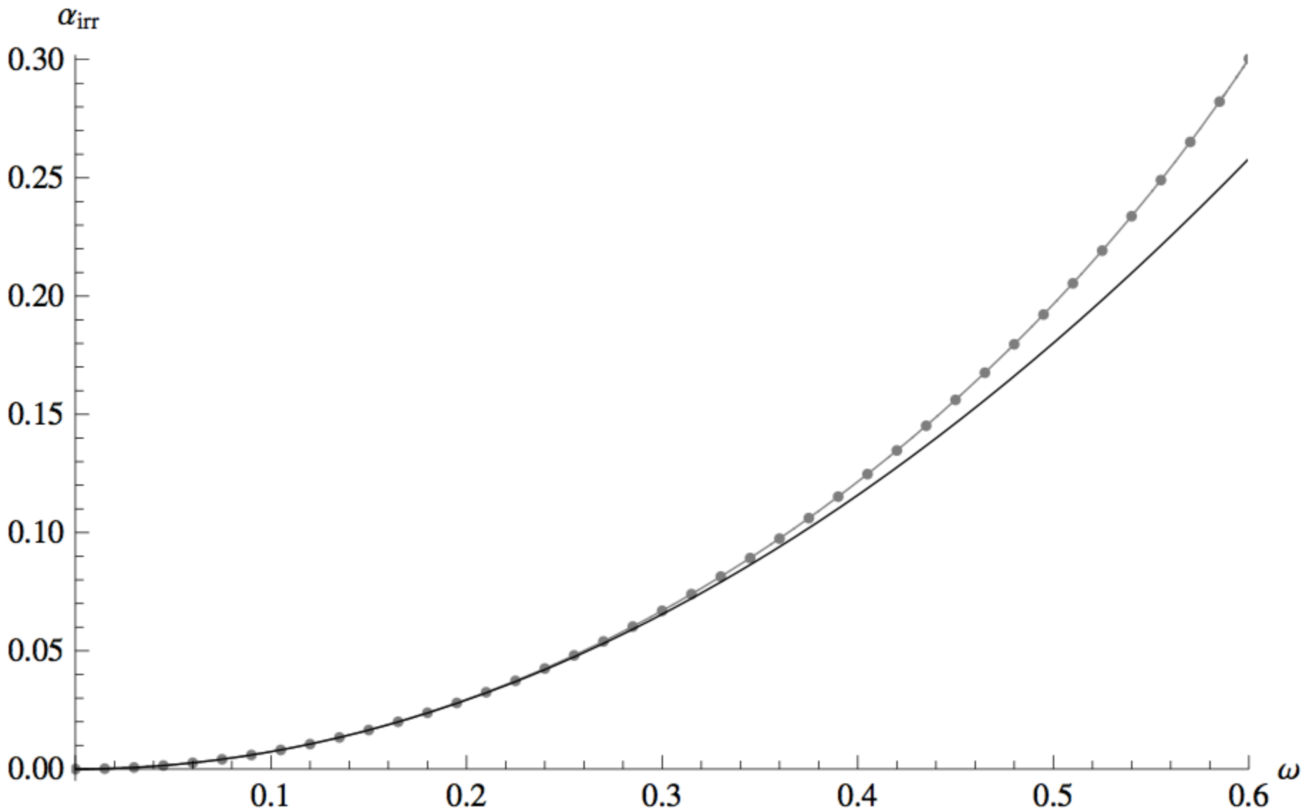


Figure 1: The figure depicts the monodromy around the irregular singular point α_{irr} as a function of the frequency ω for fixed values of the mass $M = 0.7$, angular momenta $a = 0.2$ and $l = m = 2$. The *black* line outlines the analytical perturbative results for α_{irr} while the *gray* line represents the fit of the numerical data, the *gray dots*.

Schw. BH

In Schwarzschild coordinates we have

$$ds^2 = -\left(1+\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \ ,$$

Solution to the KG equation

$$\psi(t,r,\theta,\phi) = e^{-i\omega t + im\phi}R(r)S(\theta) \ ,$$

A set of ODEs

1) $$\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) - \frac{m^2}{\sin^2\theta}\right]S(\theta) = -K_\ell S(\theta) \ ,$$

2) $$\left[\partial_r\, r\,(r-2M)\partial_r + \frac{(2M)^3\omega^2}{(r-2M)} + (r^2 + 2M(r+2M))\omega^2\right]R(r) = K_\ell R(r) \ .$$

The singular points of the ODE for Schwarzschild corresponds to the coordinate singularity r = 0, the horizon r = 2M, and infinity r → ∞.

$$R(r) = (r-2M)^{i\alpha_+}(a_1+\ldots) + (r-2M)^{-i\alpha_+}(a_2+\ldots) \ , \qquad \alpha_+ = 2M\omega \ ,$$
$$R(r) = r^{i\alpha_-}(b_1 + b_3\log r + \cdots) + r^{-i\alpha_-}(b_2+\cdots) \ , \qquad \alpha_- = 0 \ .$$

The singularity at r → ∞ is an irregular singular point (of rank 1) and the problem of finding αirr will be analogous to that of Kerr

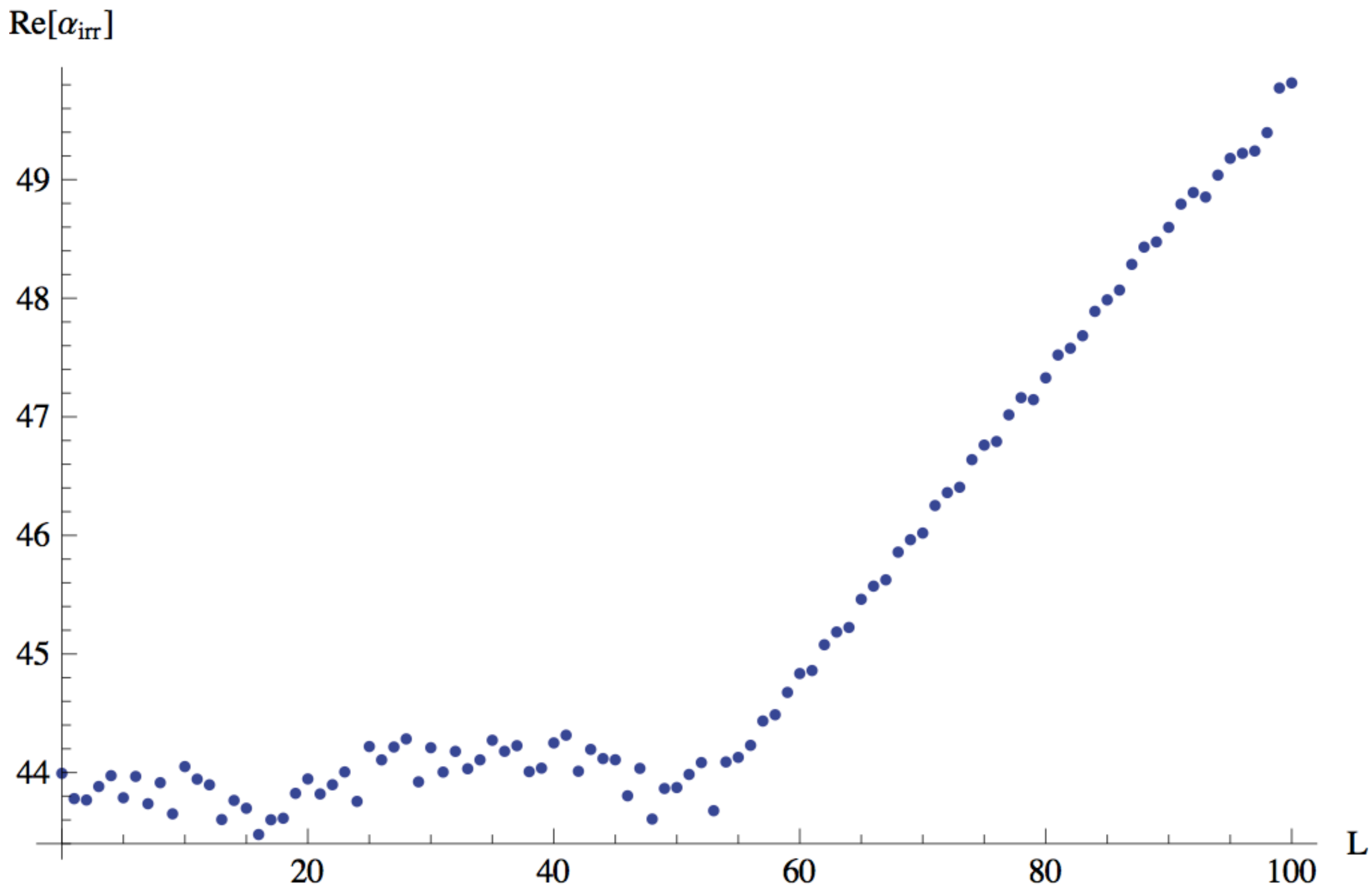


Fig. 2 The figure shows the real part of the monodromy around infinity as a function of the eigenvalue L in KL=L(L+1) for ω=Log[3]/(4 π) + 399 i /4 . We note that for large L>>1 and any ω=Log[3]/(4 π) + (2n-1)i /4 the monodromy behaves linearly with L.

Final Remarks

Every good story has to have a little bit of drama, our computation has that!

As in any other scattering problem, the phase solution should have a definite sign as the boundary is approached to correspond to waves falling into or vout of the horizon.

The ingoing/outgoing solutions at horizons are often written in so-called "tortoise coordinates" r*, where they take the form of plane waves:

$$\psi \sim e^{i\omega r_*}(\ldots) + e^{-i\omega r_*}(\ldots) \ ,$$

✓ For the outer horizon, adopting the left-eigenvectors of M+ as a basis coincidences with ingoing/outgoing boundary conditions

✗ At infinity, the plane wave basis of the form above diagonalizes the formal monodromy e2πiΛ_0 .Hence to find that base we have to go through the intermediate basis of M∞.

The computation of grey body factors (i.e. scattering coefficients) boils down to finding the connection matrix that maps the two basis of solutions.

$$\Phi_+ = \Phi_\infty \mathcal{M}_{\infty\to+} \,.$$

where

$$\mathcal{M}_{\infty\to+} \sim \begin{pmatrix} \sinh\pi(\alpha_{irr}-\alpha_+ +\alpha_-) & \sinh\pi(\alpha_{irr}+\alpha_+ +\alpha_-) \\ \sinh\pi(\alpha_{irr}+\alpha_+ -\alpha_-) & \sinh\pi(\alpha_{irr}-\alpha_+ -\alpha_-) \end{pmatrix} ;$$

$$\Phi_\infty = \Phi_{\mathrm{pw}} \mathcal{M}_{\mathrm{pw}\to\infty} \,,$$

where

$$\mathcal{M}_{\mathrm{pw}\to\infty} \sim \begin{pmatrix} e^{\pi\alpha_{irr}} & e^{-\pi\alpha_{irr}} \\ \sinh\pi(\lambda-\alpha_{irr}) & \sinh\pi(\lambda+\alpha_{irr}) \end{pmatrix} ,$$

$$\mathcal{M}_{\mathrm{pw}\to+} = \mathcal{M}_{\mathrm{pw}\to\infty}\mathcal{M}_{\infty\to+} \,,$$

The normalizations are important!

The caveat that we have to determine the normalization of the solutions to make the product of both connection matrices meaningful.

$$\mathcal{M}_{\mathrm{pw}\to+} = \begin{pmatrix} d_1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathcal{T}_1} & \frac{\mathcal{R}_1}{\mathcal{T}_1} \\ \frac{\mathcal{R}_1}{\mathcal{T}_1} & \frac{1}{\mathcal{T}_1} \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & d_2^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathcal{T}_2} & \frac{\mathcal{R}_2}{\mathcal{T}_2} \\ \frac{\mathcal{R}_2}{\mathcal{T}_2} & \frac{1}{\mathcal{T}_2} \end{pmatrix} \begin{pmatrix} d_3 & 0 \\ 0 & d_3^{-1} \end{pmatrix},$$

Greybody factors

$$|\mathcal{T}|^2 = 1 - |\mathcal{R}|^2 = \frac{(1-\mathcal{R}_1^2)(1-\mathcal{R}_2^2)}{(d_2^2 + \mathcal{R}_1\mathcal{R}_2)(d_2^{-2} + \mathcal{R}_1\mathcal{R}_2)}$$

$$\mathcal{R}_1 = \sqrt{e^{-2\pi\alpha_{irr}} \frac{\sinh \pi(\lambda - \alpha_{irr})}{\sinh \pi(\lambda + \alpha_{irr})}},$$

$$\mathcal{R}_2 = \sqrt{\frac{\sinh \pi(\alpha_+ + \alpha_- + \alpha_{irr}) \sinh \pi(\alpha_+ - \alpha_- + \alpha_{irr})}{\sinh \pi(\alpha_+ + \alpha_- - \alpha_{irr}) \sinh \pi(\alpha_+ - \alpha_- - \alpha_{irr})}},$$

$$\mathcal{T}_i^2 = 1 - \mathcal{R}_i^2 \quad \text{for} \quad i = 1, 2.$$

The above does not imply that we have achieved a decoupling between the horizon dynamics and the asymptotically flat geometry. Ideally we would like to conclude that M∞→+ is independent of Mpw→∞, or at least in a limit.

- The normalization of Φ∞, encoded in d2, affects the greybody factors. This normalization is not simply a constant, it will depend non-trivial on e.g. the frequency ω and mass of the BH.

- αirr does depend on all data in the ODE. So even if we ignored the role of d2 in the computation, we would have to also argue that αirr is independent of the horizon data (α±) to have the physics of the horizon being independent of the dynamics at infinity.

Quasi-normal modes

A quasinormal mode (QNM) is defined as a solution that is purely ingoing at the horizon r = r+ and purely outgoing at infinity.

For real frequencies, it is not possible to satisfy these boundary conditions, so we now relax that condition and allow $\omega \in C$

$$\mathcal{M}^{\rm QNM}_{\rm pw\to +} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\mathcal{R}'}{\mathcal{T}'} & \frac{1}{\mathcal{T}'} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\mathcal{T}'} \end{pmatrix}, \qquad \mathcal{T}' \neq 0\,.$$

Explicitly

$$\mathcal{M}_{\rm pw\to +} = \begin{pmatrix} \frac{d_1 d_3 (d_2^2 + \mathcal{R}_1 \mathcal{R}_2)}{d_2 \mathcal{T}_1 \mathcal{T}_2} & \frac{d_1 (\mathcal{R}_1 + d_2^2 \mathcal{R}_2)}{d_2 d_3 \mathcal{T}_1 \mathcal{T}_2} \\ \frac{d_2 d_3 (\mathcal{R}_1 + d_2^{-2} \mathcal{R}_2)}{d_1 \mathcal{T}_1 \mathcal{T}_2} & \frac{d_2 (d_2^{-2} + \mathcal{R}_1 \mathcal{R}_2)}{d_1 d_3 \mathcal{T}_1 \mathcal{T}_2} \end{pmatrix}.$$

$$d_2^{-2} + \mathcal{R}_1 \mathcal{R}_2 = 0\,. \qquad\qquad d_2^2 + \mathcal{R}_1 \mathcal{R}_2 = 0\,,$$

The QNM are determined by the two last relations.

We have studied the scattering process of a scalar field by a black hole space-time. But our results can be extended to vector and tensor perturbations, as well as a more general cases such as Ads or dS BH scattering.

We accomplished this by analyzing the singularities of the well known Teukolsky wave equation we show how the corresponding monodromies around these points are related to scattering coefficients.

We probe into the black hole scattering by using new analytical and numerical methods which are in good agreement with the previous known results.

These observations, valid in full generality, provide new insights into the properties of the scattering coefficients and their link to the global information of the wave equation encoded in the monodromies.

Thanks.