Yangians In Deformed Super Yang-Mills Theories

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Outline

Symmetry Algebra

- $\mathcal{N} = 4$, $D = 4$ Super Yang-Mills has PSU(2,2|4) superalgebra.
- A Yangian extension exists in the planar limit of the SU(N) gauge group.
- Marginal deformations have been used to deform to $\mathcal{N} = 1$, $D = 4$ SYM theories, $\mathrm{SU}(2,2|1)\times\mathrm{U}(1)\times\mathrm{U}(1)$.
- Deformations (beta and twists) have been shown to maintain integrability.
- Twisted theories have a non-standard coproduct.

Marginal Deformation

We break the $\mathcal{N} = 4$ to a $\mathcal{N} = 1$ superconformal theory by the addition of the marginal deformation with the superpotential

$$\mathcal{W} = ih\mathrm{Tr}\left(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right) + \frac{ih'}{3}\mathrm{Tr}\left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3\right).$$

For an exact marginal deformation,

$$|h|^2\left(1 + \frac{1}{N}\left(e^{i\pi\beta} - e^{-i\pi\beta}\right)^2\right) + |h'|^2\frac{N^2 - 4}{2N^2} = g^2.$$

We left $h' = 0$ and β real in the large N limit. Then $h = g$.

The Deformed Lagrangian

The Lagrangian of a deformed $\mathcal{N} = 4$, $D = 4$ SYM:

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\mathcal{D}^\mu\bar{\Phi}^i)(\mathcal{D}_\mu\Phi_i) - \frac{1}{2}[\Phi_i,\Phi_j]_{C_{ij}}[\bar{\Phi}^i,\bar{\Phi}^j]_{C_{ij}} \right.$$

$$+\frac{1}{4}[\Phi_i,\bar{\Phi}^i][\Phi_j,\bar{\Phi}^j] + \lambda_A\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^A - \mathrm{i}[\lambda_4,\lambda_i]_{B_{4i}}\bar{\Phi}^i$$

$$\left. +\mathrm{i}[\bar{\lambda}^4,\bar{\lambda}^i]_{B_{4i}}\Phi_i + \frac{\mathrm{i}}{2}\epsilon^{ijk}[\lambda_i,\lambda_j]_{B_{ij}}\Phi_k + \frac{\mathrm{i}}{2}\epsilon_{ijk}[\bar{\lambda}^i,\bar{\lambda}^j]_{B_{ij}}\bar{\Phi}^k\right)$$

where $[\Phi_i,\Phi_j]_{C_{ij}} = e^{\mathrm{i}C_{ij}}\Phi_i\Phi_j - e^{-\mathrm{i}C_{ij}}\Phi_j\Phi_i$ and $[\lambda_A,\lambda_B]_{B_{AB}} = e^{\mathrm{i}B_{AB}}\lambda_A\lambda_B - e^{-\mathrm{i}B_{AB}}\lambda_B\lambda_A$. Here $1 \le i,j \le 3$ and $1 \le A,B \le 4$.

SU(2|3)

- SU(2|3) sector, a subset of states of the PSU(2, 2|4) theory.
- The field content is: $\Phi_J = \{\phi_1, \phi_2, \phi_3; \psi_1, \psi_2\}$.
  - $|\phi_a\rangle = c_a^\dagger c_4^\dagger |0\rangle$
  - $|\psi_\alpha\rangle = a_\alpha^\dagger c_4^\dagger |0\rangle$
- The generators of the SU(2|3) superalgebra are

$$R^a{}_b = c_b^\dagger c^a - \tfrac{1}{3}\delta_b^a c_c^\dagger c^c, \quad L^\alpha{}_\beta = a_\beta^\dagger a^\alpha - \tfrac{1}{2}\delta_\beta^\alpha a_\gamma^\dagger a^\gamma,$$
$$D = c_c^\dagger c^c + \tfrac{3}{2} a_\gamma^\dagger a^\gamma, \quad S^\gamma{}_c = c_c^\dagger a^\gamma, \quad Q^c{}_\gamma = a_\gamma^\dagger c^c.$$

- Maximal subalgebra
  - Large enough for interesting structural features to arise.
  - Higher order length fluctuations.

Yangian Algebra Y(SU(2|3)): J^A, Q^A, ...

Defining relations

$$\begin{aligned} \left[J^A, J^B\right\} &= f^{AB}{}_C J^C, \\ \left[J^A, Q^B\right\} &= f^{AB}{}_C Q^C, \\ \left[Q^{[A}, \left[Q^B, J^{C\}}\right\}\right\} &= \alpha f^{AG}{}_D f^{BH}{}_E f^{CK}{}_F f_{GHK} J^{\{D} J^E J^{F\}} \end{aligned}$$

The J^A are SU(2|3) generators. The tree-level, first nonlocal Yangian generator is

$$Q_0^A = -f^A{}_{CB} \sum_{i<j} J_0^B(i) J_0^C(j).$$

Standard Coproducts

A coproduct is a holomorphic map $\Delta : \mathcal{A} \to \mathcal{A}$. Introduces the idea of single site, double site, etc. . . representations for the algebra $\mathcal{A}$. The coproduct for the ordinary SU(2|3) generators

$$\Delta J^A = J^A \otimes 1 + 1 \otimes J^A.$$

and the coproduct to create the two-site Yangian generators

$$\begin{aligned} \Delta Q^A = \; & Q^A \otimes 1 + 1 \otimes Q^A \\ & - f^A{}_{CB} J^B \otimes J^C \end{aligned}$$

will be used to construct tree level representations.

Hamiltonian

The two-site Hamiltonian in terms of oscillators is

$$\begin{aligned} H(1,2) = \; & \left(c_a^\dagger(1)c_b^\dagger(2) - c_b^\dagger(1)c_a^\dagger(2)\right) c^b(2)c^a(1) \\ & + \left(c_a^\dagger(1)a_\alpha^\dagger(2) + a_\alpha^\dagger(1)c_a^\dagger(2)\right) a^\alpha(2)c^a(1) \\ & + \left(a_\alpha^\dagger(1)c_a^\dagger(2) + c_a^\dagger(1)a_\alpha^\dagger(2)\right) c^a(2)a^\alpha(1) \\ & + \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) + a_\beta^\dagger(1)a_\alpha^\dagger(2)\right) a^\beta(2)a^\alpha(1). \end{aligned}$$

The commutation relations of the oscillators are $\{c^a(i), c_b^\dagger(j)\} = \delta_b^a \delta_{ij}$ and $[a^\alpha(i), a_\beta^\dagger(j)] = \delta_\beta^\alpha \delta_{ij}$.

Quadractic Casimir

The quadratic Casimir of the subalgebra SU(2|3) is

$$g_{AB}J^A J^B = \frac{1}{3}D^2 + \frac{1}{2}L^{\alpha}{}_{\beta}L^{\beta}{}_{\alpha} - \frac{1}{2}R^a{}_b R^b{}_a - \frac{1}{2}[Q^c{}_{\gamma}, S^{\gamma}{}_c].$$

A Casimir of any algebra has the property

$$[g_{AB}J^A J^B, J^C] = 0$$

When acting on any two-particle state $|\Phi_I \Phi_J\rangle$ the quadratic Casimir and the two-site Hamiltonian are equivalent,

$$H(1,2)|\Phi_I \Phi_J\rangle = g_{AB}J^A J^B |\Phi_I \Phi_J\rangle.$$

Eigenstates

The two-particle eigenstates of the Hamiltonian form two towers, 13 symmetric ($H_{12}|\Phi_1\Phi_2\rangle_+ = 0 \cdot |\Phi_1\Phi_2\rangle_+$) and 12 antisymmetric ($H_{12}|\Phi_1\Phi_2\rangle_- = 2|\Phi_1\Phi_2\rangle_-$)) eigenstates.

$$\begin{aligned}
|ab\rangle_\pm &= -\left(c_a^\dagger(1)c_b^\dagger(2) \pm c_b^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|a\beta\rangle_\pm &= \left(c_a^\dagger(1)a_\beta^\dagger(2) \mp a_\beta^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|\alpha\beta\rangle_\pm &= \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) \mp a_\beta^\dagger(1)a_\alpha^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle.
\end{aligned}$$

Analysis of Yangian Symmetry

A property of the dilatation generator is $[D, J^A] = (\text{dim}J^A)J^A$. Also, $[D, Q^A] = (\text{dim}J^A)Q^A$. Expanding both the dilatation generator and the Yangian,

$$[D, Q^A] = [D_0 + g_{\text{YM}}^2 D_2 + \cdots, Q_0^A + g_{\text{YM}} Q_1^A + g_{\text{YM}}^2 Q_2^A + \cdots].$$

Group in powers of the Yang-Mills coupling to $\mathcal{O}(g^2)$

$$(\text{dim}J^A)(Q_0^A + g_{\text{YM}} Q_1^A + g_{\text{YM}}^2 Q_2^A) + g_{\text{YM}}^2 [D_2, Q_0^A] \approx (\text{dim}J^A)Q^A.$$

That means $[D_2, Q_0^A]$ must give zero (or approximately). In the large N (planar limit) D_2 is our spin chain Hamiltonian, H.

Edge Effects

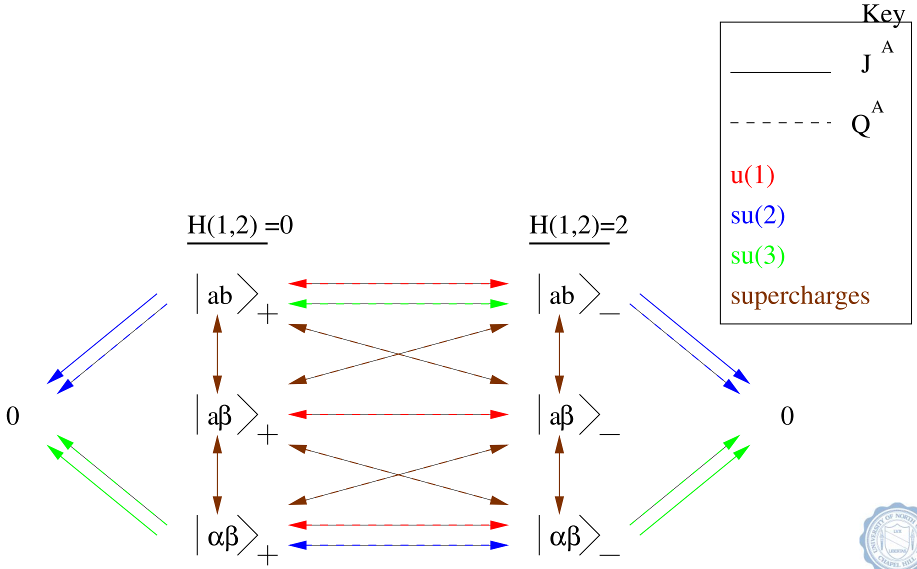
In PSU$(2,2|4)$, an explicit check of the commutator gives the lattice derivative or 'edge effects' of the system, $[D_2, Q_0^A] = q^A \sim 0$, $q_{1L}^A = J^A(1) - J^A(L)$. Introduce the identity,

$$Q_{12}^A = \frac{1}{4}\left[g_{BC}J(1)^B J(2)^C, q_{12}^A\right],$$

Recall, the quadratic Casimir is equivalent to the Hamiltonian when acting on states. For the two-site case,

$$\begin{aligned}\left[H(1,2), Q_{12}^A\right]|\Phi_I\Phi_J\rangle &= \tfrac{1}{4}\left[H(1,2)^2 q_{12}^A + q_{12}^A H(1,2)^2 \right.\\ &\quad \left. -2H(1,2)q_{12}^A H(1,2)\right]|\Phi_I\Phi_J\rangle.\\ &= q_{12}^A|\Phi_I\Phi_J\rangle\end{aligned}$$

Yangian on Two-Particle States



Deformed Hamiltonian

The deformed R matrix, a solution to the Yang-Baxter equation is

$$\tilde{R}^{KL}_{IJ}(u) = \frac{1}{u+i}\left(ue^{-iB_{IJ}}\mathcal{I}^{KL}_{IJ} + i\mathcal{P}^{KL}_{IJ}\right).$$

The identity and projection operators are $\mathcal{I}^{KL}_{IJ} = \delta^K_I \delta^L_J$ and $\mathcal{P}^{KL}_{IJ} = \delta^L_I \delta^K_J$. The deformed monodromy matrix is

$$\tilde{\mathcal{T}}^{J;\beta_1\ldots\beta_L}_{I;\alpha_1\ldots\alpha_L} = \tilde{R}^{b_{L-1}\beta_L}_{I\alpha_L}\tilde{R}^{b_{L-2}\beta_{L-1}}_{b_{L-1}\alpha_{L-1}}\cdots\tilde{R}^{b_1\beta_2}_{b_2\alpha_2}\tilde{R}^{J\beta_1}_{b_1\alpha_1}\exp\left[i\pi\sum_{i=1}^{L}\sum_{j=1}^{i-1}([\alpha_i]+[\beta_i])[\alpha_j]\right]$$

The deformed transfer matrix is $\tilde{\mathcal{T}}(u) = (-)^{[J]}\tilde{\mathcal{T}}^J_J(u)$. The deformed Hamiltonian is derived as

$$\tilde{\mathcal{H}} = -i\left(\tilde{\mathcal{T}}(u)\right)^{-1}\frac{d}{du}\tilde{\mathcal{T}}(u)\Big|_{u=0}$$

Deformed Two-Site Hamiltonian

The two-site transfer matrix is

$$\tilde{\mathcal{T}}(u) = \tilde{R}^{b_1\beta_2}_{a\alpha_2}\tilde{R}^{a\beta_1}_{b_1\alpha_1}\exp\left[i\pi([\alpha_2]+[\beta_2])[\alpha_1]\right].$$

Using the prescribed method, we derive the deformed Hamiltonian to be

$$\begin{aligned}\tilde{\mathcal{H}} &= \left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} - \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}e^{-iB_{\alpha_1\alpha_2}}\right) + \left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} - \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}e^{-iB_{\alpha_2\alpha_1}}\right)\\ &= \left(\tilde{\mathcal{H}}^{\beta_1\beta_2}_{\alpha_1\alpha_2}\right) + \left(\tilde{\mathcal{H}}^{\beta_2\beta_1}_{\alpha_2\alpha_1}\right).\end{aligned}$$

Define the deformed two-site Hamiltonians, $\tilde{H}(1,2) \equiv \tilde{\mathcal{H}}^{\beta_1\beta_2}_{\alpha_1\alpha_2}$ and $\tilde{H}(2,1) \equiv \tilde{\mathcal{H}}^{\beta_2\beta_1}_{\alpha_2\alpha_1}$.

Deformed Hamiltonian

The oscillator representation of the deformed Hamiltonian

$$\begin{aligned}\tilde{H}_{12} &= \left(c_a^\dagger(1)c_b^\dagger(2) - e^{-iB_{ab}}c_b^\dagger(1)c_a^\dagger(2)\right)c^b(2)c^a(1)\\ &+ \left(c_a^\dagger(1)a_\alpha^\dagger(2) + e^{-iB_{a\alpha}}a_\alpha^\dagger(1)c_a^\dagger(2)\right)a^\alpha(2)c^a(1)\\ &+ \left(a_\alpha^\dagger(1)c_a^\dagger(2) + e^{-iB_{\alpha a}}c_a^\dagger(1)a_\alpha^\dagger(2)\right)c^a(2)a^\alpha(1)\\ &+ \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) + e^{-iB_{\alpha\beta}}a_\beta^\dagger(1)a_\alpha^\dagger(2)\right)a^\beta(2)a^\alpha(1).\end{aligned}$$

B_{IJ} is a real, antisymmetric matrix.

Problems

Due to twisting, the Hamiltonian is no longer a Casimir of the SU(2|3) superalgebra

$$\left[\tilde{H}(1,2), J_{12}^{A}\right] \neq 0,$$

nor do we generally have the edge effects

$$\left[\tilde{H}(1,2), Q_{12}^{A}\right] \neq q_{12}^{A},$$

when using the previous (coproduct) construction for J_{12}^{A} and Q_{12}^{A}.

Reshitikhin Twist

The Reshitikhin twist is generated by a deforming function F. The R-matrix deforms as

$$\tilde{R}(u) = FR(u)F^{-1}.$$

The coproduct receives a deformation as well,

$$\Delta^{(F)} = F\Delta F^{-1}.$$

The deforming function has the definition

$$F = exp\left[\frac{i}{2}\sum_{I<J} B_{IJ}\left(E^{II}\otimes E^{JJ} - E^{JJ}\otimes E^{II}\right)\right]$$

Twisted Coproducts

Twisted coproducts $\Delta J^A{}_B = K_{AB} \otimes J^A{}_B + J^A{}_B \otimes K_{BA}$.

$$\begin{aligned}
\Delta R^a{}_b &= K_{ab} \otimes R^a{}_b + R^a{}_b \otimes K_{ba}, \\
\Delta L^\alpha{}_\beta &= K_{\alpha\beta} \otimes L^\alpha{}_\beta + L^\alpha{}_\beta \otimes K_{\beta\alpha}, \\
\Delta Q^c{}_\gamma &= K_{c\gamma} \otimes Q^c{}_\gamma + Q^c{}_\gamma \otimes K_{\gamma c}, \\
\Delta S^\gamma{}_c &= K_{\gamma c} \otimes S^\gamma{}_c + S^\gamma{}_c \otimes K_{c\gamma}, \\
\Delta D &= 1 \otimes D + D \otimes 1.
\end{aligned}$$

The twisting is brought about by

$$K_{IJ} = \exp\left[\frac{i}{2}\sum_{K=1}^{5}\left(B_{IK} - B_{JK}\right)E_{KK}\right].$$

We define matrices $(E_{IJ})_{KL} = \delta_{IK}\delta_{JL}$ which satisfy $[E_{IJ}, E_{KL}] = \delta_{KJ}E_{IL} - \delta_{IL}E_{AJ}$. The E_{IJ} are the generators of U(2|3).

Yangians Via Twisted Coproducts

The first Yangian construction via coproducts is

$$\begin{aligned}\Delta Q^I{}_J \quad &\sim \quad K_{IJ} \otimes Q^I{}_J + Q^I{}_J \otimes K_{JI} \\ &+\tfrac{1}{2} \textstyle\sum_{K=1}^{5} \left(J^I{}_K K_{KJ} \otimes K_{KI} J^K{}_J - K_{IK} J^K{}_J \otimes J^I{}_K K_{JK} \right) .\end{aligned}$$

An example two-site Yangian is

$$\begin{aligned}\Delta Q_{(R^a_b)}{}^a{}_b \quad = \quad & K_{ab} \otimes Q_{(R^a_b)}{}^a{}_b + Q_{(R^a_b)}{}^a{}_b \otimes K_{ba} \\ &+\tfrac{1}{2} \left(R^a{}_c K_{cb} \otimes K_{ca} R^c{}_b - K_{ac} R^c{}_b \otimes R^a{}_c K_{bc} \right) \\ &+\tfrac{1}{2} \left(Q^a{}_\gamma K_{\gamma b} \otimes K_{\gamma a} S^\gamma{}_b + K_{a\gamma} S^\gamma{}_b \otimes Q^a{}_\gamma K_{b\gamma} \right) \\ &-\tfrac{1}{6} \delta^a_b \left(Q^c{}_\gamma K_{\gamma c} \otimes K_{\gamma c} S^\gamma{}_c + K_{c\gamma} S^\gamma{}_c \otimes Q^c{}_\gamma K_{c\gamma} \right)\end{aligned}$$

It Works

Acting on two particle states, the deformed Hamiltonian is equivalent to the (deformed) Casimir

$$\Delta J^A{}_B \Delta J^B{}_A |\Phi_I \Phi_J\rangle = \tilde{H}(1,2)|\Phi_I \Phi_J\rangle.$$

We check the one-loop calculation of the dilatation generator again,

$$\left[\tilde{H}(1,2), Q^A_{12\,B}\right] = \tilde{q}^A_{12\,B},$$

where the edge effect term, $\tilde{q}^A_{12\,B}$, has a deformation dependence

$$\tilde{q}^A_{12\,B} = J^A{}_B \otimes K_{AB} - K_{BA} \otimes J^A{}_B.$$

We find for an infinite length spin chain when $J^A_B(1), J^A_B(L) \rightarrow 0$, then $\tilde{q}^A{}_B \rightarrow 0$. Recall, this is what we want!

Deformations Maintaining $\mathcal{N} = 1$ SCFT: Case 1

Residual SU(2)×U(1)^3 symmetry. This is the beta deformation of Lunin-Maldacena. $B_{13} = B_{21} = B_{32} = \gamma$.

Residual Symmetry	Remaining Symmetry
$\Delta L^{\alpha}{}_{\beta} = 1 \otimes L^{\alpha}{}_{\beta} + L^{\alpha}{}_{\beta} \otimes 1,$	$\Delta R^{a}{}_{b} = K_{ab} \otimes R^{a}{}_{b} + R^{a}{}_{b} \otimes K_{ba},$
$\Delta D = 1 \otimes D + D \otimes 1,$	$\Delta Q^{c}{}_{\gamma} = K_{c\gamma} \otimes Q^{c}{}_{\gamma} + Q^{c}{}_{\gamma} \otimes K_{\gamma c},$
$\Delta R^{c}{}_{c} = 1 \otimes R^{c}{}_{c} + R^{c}{}_{c} \otimes 1,$	$\Delta S^{\gamma}{}_{c} = K_{\gamma c} \otimes S^{\gamma}{}_{c} + S^{\gamma}{}_{c} \otimes K_{c\gamma}.$

Deformations Maintaining $\mathcal{N} = 1$ SCFT: Case 2

Residual SU(2|1)×U(1)^2 symmetry.

$B_{12} = B_{13} = B_{23} = B_{1\alpha} = -B_{2\alpha} = \gamma$.

Residual Symmetry

$$\Delta L^{\alpha}{}_{\beta} = 1 \otimes L^{\alpha}{}_{\beta} + L^{\alpha}{}_{\beta} \otimes 1,$$
$$\Delta Q^{3}{}_{\gamma} = 1 \otimes Q^{3}{}_{\gamma} + Q^{3}{}_{\gamma} \otimes 1,$$
$$\Delta S^{\gamma}{}_{3} = 1 \otimes S^{\gamma}{}_{3} + S^{\gamma}{}_{3} \otimes 1,$$
$$\Delta D = 1 \otimes D + D \otimes 1,$$
$$\Delta R^{c}{}_{c} = 1 \otimes R^{c}{}_{c} + R^{c}{}_{c} \otimes 1,$$

Remaining Symmetry

$$\Delta R^{a}{}_{b} = K_{ab} \otimes R^{a}{}_{b} + R^{a}{}_{b} \otimes K_{ba},$$
$$\Delta Q^{c}{}_{\gamma} = K_{c\gamma} \otimes Q^{c}{}_{\gamma} + Q^{c}{}_{\gamma} \otimes K_{\gamma c},$$
$$\Delta S^{\gamma}{}_{c} = K_{\gamma c} \otimes S^{\gamma}{}_{c} + S^{\gamma}{}_{c} \otimes K_{c\gamma}.$$

Conclusion

- Constructed a Yangian algebra for SU(2|3)
- The deformed theory corresponds to a deformed R-matrix
- Introduced twisted coproducts corresponding to the twisted R-matrix
- Showed that twisted coproducts could be used to describe broken symmetry in $\mathcal{L}$
- The deformed Hamiltonian (one-loop dilatation generator) still has SU(2|3) Yangian symmetry but with twisted coproduct

Yangians Via Twisted Coproducts

$$
\begin{aligned}
\Delta Q_{(R^a_b)}{}^a{}_b &= K_{ab} \otimes Q_{(R^a_b)}{}^a{}_b + Q_{(R^a_b)}{}^a{}_b \otimes K_{ba} \\
&+\tfrac{1}{2} h \left(R^a{}_c K_{cb} \otimes K_{ca} R^c{}_b - K_{ac} R^c{}_b \otimes R^a{}_c K_{bc} \right) \\
&+\tfrac{1}{2} h \left(Q^a{}_\gamma K_{\gamma b} \otimes K_{\gamma a} S^\gamma{}_b + K_{a\gamma} S^\gamma{}_b \otimes Q^a{}_\gamma K_{b\gamma} \right) \\
&-\tfrac{1}{6} h \delta^a_b \left(Q^c{}_\gamma K_{\gamma c} \otimes K_{\gamma c} S^\gamma{}_c + K_{c\gamma} S^\gamma{}_c \otimes Q^c{}_\gamma K_{c\gamma} \right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta Q_{(L^\alpha_\beta)}{}^\alpha{}_\beta &= K_{\alpha\beta} \otimes Q_{(L^\alpha_\beta)}{}^\alpha{}_\beta + Q_{(L^\alpha_\beta)}{}^\alpha{}_\beta \otimes K_{\beta\alpha} \\
&+\tfrac{1}{2} h \left(L^\alpha{}_\gamma K_{\gamma\beta} \otimes K_{\gamma\alpha} L^\gamma{}_\beta - K_{\alpha\gamma} L^\gamma{}_\beta \otimes L^\alpha{}_\gamma K_{\beta\gamma} \right) \\
&+\tfrac{1}{2} h \left(S^\alpha K_{c\beta} \otimes K_{c\alpha} Q^c{}_\beta + K_{\alpha c} Q^c{}_\beta \otimes S^\alpha{}_c K_{\beta c} \right) \\
&-\tfrac{1}{4} h \delta^\alpha_\beta \left(S^\gamma{}_c K_{c\gamma} \otimes K_{c\gamma} Q^c{}_\gamma + K_{\gamma c} Q^c{}_\gamma \otimes S^\gamma{}_c K_{\gamma c} \right),
\end{aligned}
$$

Yangians Via Twisted Coproducts

$$
\begin{aligned}
\Delta Q_{(Q^a_\alpha)}{}^a{}_\alpha &= K_{a\alpha} \otimes Q_{(Q^a_\alpha)}{}^a{}_\alpha + Q_{(Q^a_\alpha)}{}^a{}_\alpha \otimes K_{\alpha a} \\
&\quad + \tfrac{1}{2} h \left(Q^a{}_\gamma K_{\gamma\alpha} \otimes K_{\gamma a} L^\gamma{}_\alpha - K_{a\gamma} L^\gamma{}_\alpha \otimes Q^a{}_\gamma K_{\alpha\gamma} \right) \\
&\quad + \tfrac{1}{2} h \left(R^a{}_c K_{c\alpha} \otimes K_{ca} Q^c{}_\alpha - K_{ac} Q^c{}_\alpha \otimes R^a{}_c K_{\alpha c} \right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta Q_{(S^\alpha_a)}{}^\alpha{}_a &= K_{\alpha a} \otimes Q_{(S^\alpha_a)}{}^\alpha{}_a + Q_{(S^\alpha_a)}{}^\alpha{}_a \otimes K_{a\alpha} \\
&\quad + \tfrac{1}{2} h \left(S^\alpha{}_c K_{ca} \otimes K_{ca} R^c{}_a - K_{\alpha c} R^c{}_a \otimes S^\alpha{}_c K_{ac} \right) \\
&\quad + \tfrac{1}{2} h \left(L^\alpha{}_\gamma K_{\gamma a} \otimes K_{\gamma\alpha} S^\gamma{}_a - K_{\alpha\gamma} S^\gamma{}_a \otimes L^\alpha{}_\gamma K_{a\gamma} \right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta Q_{(D)} &= 1 \otimes Q_{(D)} + Q_{(D)} \otimes 1 \\
&\quad + \tfrac{1}{4} h \left(S^\gamma{}_c K_{c\gamma} \otimes K_{c\gamma} Q^c{}_\gamma + K_{\gamma c} Q^c{}_\gamma \otimes S^\gamma{}_c K_{\gamma c} \right).
\end{aligned}
$$