On the duality between CS-matter theory and strings in $AdS_4 \times \mathbb{CP}^3$: loops vs. spins

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based on work with T. McLoughlin and A. Tseytlin
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Many reasons to study 3d CFT-s:

- potential revelations on the M2-brane theory

  attempts by Bagger, Lambert

- fixed points of condensed matter systems

- understanding of part of the landscape of $d = 4$ string vacua

- potentially tractable examples of gauge/string duality

On the M2-brane theory

- AdS/CFT: theory is conformal and dual to M-theory on $AdS_4 \times S^7$
  - ◇ fixed point of the D2 brane theory
    - – 8 physical scalars
    - – perhaps additional, topological degrees of freedom
  - ◇ 3d gauge theory has dimensionful coupling $\mapsto$ must disappear at the fixed point $\mapsto$ only CS-type quadratic term
  - ◇ Parameters: 't Hooft coupling: $\lambda = g^2_{\text{YM}} N \ \mapsto \lambda_{\text{CS}} = \frac{N}{k_{\text{CS}}}$
  - ◇ Interpretation of level k_{CS}? Natural values? 10d connection?

Outline

- The $\mathcal{N} = 6$ CS-matter theory
- The conjectured Bethe ansatz and its relation to $AdS_5 \times S^5$
- Worldsheet calculations, comparison and differences
- Outlook

$U(N) \times U(N)$ Chern-Simons-matter theory with $\mathcal{N} = 6$ susy

– special case of $\mathcal{N} = 3$ construction

- $SO(6) \simeq SU(4)$ R-symmetry
- 4 complex scalar fields: $Y^A \in \mathbf{N} \times \bar{\mathbf{N}}$ and $Y_A^\dagger \in \bar{\mathbf{N}} \times \mathbf{N}$
- 4 complex fermions
- supermultiplet: scalars in $\mathbf{4}$ and fermions in $\bar{\mathbf{4}}$ $\mapsto$ susy gen's in $\mathbf{6}$

$$
\begin{aligned}
S \;=\; & \frac{k_{\text{cs}}}{4\pi} \int d^3x \text{Tr} \left[\epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho - \hat{A}_\mu \partial_\nu \hat{A}_\rho - \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \right. \\
& + D_\mu Y_A^\dagger D^\mu Y^A + \frac{1}{12} Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12} Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\
& \left. - \frac{1}{2} Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + \frac{1}{3} Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger + \text{fermions} \right]
\end{aligned}
$$

- superpotential $W = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \text{Tr} \, [A_a B_{\dot{a}} A_b B_{\dot{b}}]$; $\; Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger)$
- Covariant derivative: $D_\mu Y^A = \partial_\mu Y^A + A_\mu Y^A - Y^A \hat{A}_\mu$

$$
\begin{aligned}
S \;=\; & \frac{k_{\rm CS}}{4\pi}\int d^3x \,\mathrm{Tr}\,\Big[\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho - \frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho) \\
& +D_\mu Y_A^\dagger D^\mu Y^A + \frac{1}{12}Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12}Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger \\
& -\frac{1}{2}Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + \frac{1}{3}Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger + \text{fermions}\Big]
\end{aligned}
$$

- Power-counting renormalizable; special choice of levels $k_1 = -k_2$

• Planar perturbation theory: Series expansion in λ^2 rather than λ

(a feature of 3d perturbation theory)

- Argued to have exact conformal invariance – $OSp(6|4)$ symmetry

...; Gaiotto, Yin;...

• Theory constructible from $\mathcal{N} = 4$ $d = 2 + 1$ SYM theory broken to $\mathcal{N} = 3$ and deformed by supersmmetric CS term and flown to $E \ll m = g^2_{\rm YM} k_{\rm CS}/(4\pi)$

Aharony, Bergman, Jafferis, Maldacena

String/M-theory dual: almost-max susy, correct symmetries

- $AdS_4 \times \mathbb{CP}^3$ has $SO(3,2) \times SO(6) \simeq Sp(4) \times SO(6)$ symmetry
- $\mathbb{Z}_k$ orbifold projection of $AdS_4 \times S^7$ on nonsingular fiber

$$\begin{array}{ccc} S^1 & \rightarrow & S^7 \\ & & \downarrow \\ & & \mathbb{CP}^3 \end{array}$$

- M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ (weak coupling stability ensured by supersymmetry)
- string theory limit: $k \to \infty$ relate k and k_{CS}

$$ds^2_{AdS_4 \times S^7} = \frac{R^4}{4}\left(ds^2_{\mathrm{AdS}_4} + 4 ds^2_{S^7}\right) \qquad F_{(4)} \propto \mathrm{Vol}(\mathrm{AdS}_4)$$

$$ds^2_{S^7} = (d\phi + \omega)^2 + ds^2_{\mathbb{CP}^3} \quad \xrightarrow{\mathbb{Z}_k} \quad ds^2 = \frac{1}{k^2}(d\phi + k\omega)^2 + ds^2_{\mathbb{CP}^3}$$

- Account for volume reduction:

$$ds^2 = \frac{R^3}{4k_{\mathrm{CS}}}\left(ds^2_{\mathrm{AdS}_4} + 4 ds^2_{\mathbb{CP}^3}\right) \qquad e^{2\phi} = \frac{R^3}{k^3_{\mathrm{CS}}}$$

$$F_2 = k_{\mathrm{CS}}\, \mathbb{J}_{\mathbb{CP}^3} \qquad F_4 = \frac{3}{8} R^3 \mathrm{Vol}_{\mathrm{AdS}_4}$$

So here is another conjectured gauge/string duality. Why bother?

◇ less-than maximal susy: may exhibit features absent in $AdS_5 \times S^5$

– different coupling constant dependence

– fewer protected quantities; more interpolating functions

◇ Tractable both at weak and strong coupling and thus testable

Where to begin?

◇ expect agreement for all quantities protected by symmetries

$\longrightarrow$ focus on unprotected quantities – e.g. anomalous dimensions

Leading order dilatation operator for scalar operators Minahan, Zarembo

- main difference from $\mathcal{N}=4$ SYM: scalars in bifundamental rep.

$\mapsto$ gauge-invariant scalar operators are of the type

$$\text{Tr}\,[Y^{A_1}Y^\dagger_{B_1}Y^{A_2}Y^\dagger_{B_2}Y^{A_3}Y^\dagger_{B_3}\ldots Y^{A_L}Y^\dagger_{B_L}]$$

- arises at 2-loops
  - has nearest and next-to-nearest neighbor interactions

$$\Gamma = \frac{\lambda^2}{2}\sum_{l=1}^{2L} H_{l,l+1,l+2}$$

$$H_{l,l+1,l+2} = \mathbb{1} - K_{l,l+1} - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}$$

- Trace and permutation operators:

$$K : V\times\bar{V} \to V\times\bar{V} \qquad K^{AB'}_{BA'} = \delta_{AB'}\delta^{BA'}$$

$$P : V\times V \to V\times V \qquad P^{AB}_{A'B'} = \delta^A_{B'}\delta^B_{A'}$$

- and the surprise is...

... that, despite the next-to-nearest neighbor interaction, this operator may be identified with a Hamiltonian derived from monodromy matrices obeying the Yang-Baxter equation and thus is integrable

– one for even sites: $T_a(u,\alpha) \propto R_{aq_1}(u) R_{a\bar{q}_1}(u+\alpha) \ldots R_{aq_L}(u) R_{a\bar{q}_L}(u+\alpha)$

– one for odd sites: $T_{\bar{a}}(u,\alpha) \propto R_{\bar{a}q_1}(u+\alpha) R_{\bar{a}\bar{q}_1}(u) \ldots R_{\bar{a}q_L}(u+\alpha) R_{\bar{a}\bar{q}_L}(u)$

- 1-loop dilatation operator is recovered by choosing $\alpha = -2$

$$\tau = \mathrm{Tr}\,[T_a] \quad \bar{\tau} = \mathrm{Tr}\,[T_{\bar{a}}] \quad [\tau, \bar{\tau}] = 0 \quad H_{\mathrm{even}} = \tau^{-1} d_u \tau \quad H_{\mathrm{odd}} = \bar{\tau}^{-1} d_u \bar{\tau}$$

Assuming all-order integrability: use machinery of discrete integrable models and symmetries preserved by the lowest dimension operator

– understand closed sectors (subsets of operators closed under RG flow)

– construct spin chain S-matrix (solve Yang-Baxter equation)

– construct Bethe ansatz $\longrightarrow$ Bethe equations

– understand coupling constant dependence

Closed sectors – should be determined by symmetries

– difference from $\mathcal{N} = 4$ SYM: both scalars and fermions are in the same representation of the R-symmetry group!

$\mapsto$ 2 scalars and 1 derivative $\Leftrightarrow$ 2 fermions

$\mapsto$ departure from familiar closed sectors

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S-matrix: vacuum Tr $[(Y^1 Y_4^\dagger)^L]$ preserves $SU(2|2) \subset OSp(6|4)$

- alternating chain $\rightarrow$ separate excitations on even and odd sites
- rep's of $SU(2|2)$; conjectured to be $(2|2)$ Ahn, Nepomechie

$Y^1 \rightarrow (Y^2, Y^3 | (\psi_3)_\alpha)$ (A-ext's) and $Y_4^\dagger \rightarrow (Y_2^\dagger, Y_3^\dagger | (\psi_2^\dagger)_\alpha)$ (B-ext's)

- 3 S-matrices: S_{AA}, S_{BB} and S_{AB}

$\uparrow$ $\uparrow$ $\uparrow$

Beisert's $psu(2|2)$ S-matrix less clear

Excitation energy: $\epsilon(p) = \sqrt{\frac{1}{4} + 4\pi^2 h^2(\lambda) \sin^2 \frac{p}{2}}$ for both

Closed sectors – should be determined by symmetries

– difference from $AdS_5 \times S^5$: both scalars and fermions are in the same representation of the R-symmetry group!

$\mapsto$ 2 scalars and 1 derivative $\Leftrightarrow$ 2 fermions

$\mapsto$ departure from $AdS_5 \times S^5$ sectors

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- 3 S-matrices: S_{AA}, S_{BB} and S_{AB}

- Formal similarity w/ S-matrices of CFT-s (e.g. Z's S-matrix for WZW) if one identifies A and B excitations with left- and right-movers.

- $(2|2)\oplus(2|2)$ excitations $\rightarrow$ formal difference with expected number of excitations on the worldsheet where there are $(8|8)$ physical fields

- The Bethe equations

Gromov, Vieira; Ahn, Nepomechie

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k}x^+_{4,j}}{1 - 1/x_{1,k}x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k}x^+_{\bar{4},j}}{1 - 1/x_{1,k}x^-_{\bar{4},j}},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{1,k} - u_{3,j} + \frac{i}{2}}{u_{1,k} - u_{3,j} - \frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x^+_{4,j}}{x_{3,k} - x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x^+_{\bar{4},j}}{x_{3,k} - x^-_{\bar{4},j}}$$

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{j\neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x^-_{4,k}x_{1,j}}{1 - 1/x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k} - x_{3,j}}{x^+_{4,k} - x_{3,j}} \times$$

$$\times \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma_{\mathrm{BES}}(u_{4,k}, u_{\bar{4},j}),$$

$$\left(\frac{x^+_{\bar{4},k}}{x^-_{\bar{4},k}}\right)^L = \prod_{j=1}^{K_{\bar{4}}} \frac{u_{\bar{4},k} - u_{\bar{4},j} + i}{u_{\bar{4},k} - u_{\bar{4},j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x^-_{\bar{4},k}x_{1,j}}{1 - 1/x^+_{\bar{4},k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{\bar{4},k} - x_{3,j}}{x^+_{\bar{4},k} - x_{3,j}} \times$$

$$\times \prod_{j\neq k}^{K_{\bar{4}}} \sigma_{\mathrm{BES}}(u_{\bar{4},k}, u_{\bar{4},j}) \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{\bar{4},k}, u_{4,j}).$$

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}} \prod_{j=1}^{K_{\bar{4}}} \frac{x^+_{\bar{4},j}}{x^-_{\bar{4},j}}$$

$$x + \frac{1}{x} = \frac{u}{h(\lambda)}$$

$$x^\pm + \frac{1}{x^\pm} = \frac{1}{h(\lambda)}\left(u \pm \frac{i}{2}\right)$$

$$p_j = \frac{1}{i} \log \frac{x^+_{4,j}}{x^-_{4,j}}$$

$$\bar{p}_j = \frac{1}{i} \log \frac{x^+_{\bar{4},j}}{x^-_{\bar{4},j}}$$

$$E = \sum_{j=1}^{K_4} \frac{1}{2}\left(\sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{p_j}{2}} - 1\right) + \sum_{j=1}^{K_{\bar{4}}} \frac{1}{2}\left(\sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{\bar{p}_j}{2}} - 1\right)$$

◇ Apparently a truncation is possible: set $K_1, K_2, K_3 = 0$; $K_4 = K_{\bar{4}}$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = -\prod_{j\neq k}^{S} \frac{u_k - u_j + i}{u_k - u_j - i} \left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\right)^2 \sigma_{BES}^2(u_k, u_j) \qquad 1 = \left(\prod_{j=1}^{S} \frac{x_j^+}{x_j^-}\right)^2$$

- Energy: $E = \sum_{j=1}^{S} \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{p_j}{2}}$ $\quad h(\lambda)^2 = \lambda^2 + \mathcal{O}(\lambda^4)$

- Suggested eq's for $SL(2)$ sector – spin S and R-charge $L = 2J$

Gromov, Vieira

◇ many similarities with Bethe eq's for the $SL(2)$ sector of $\mathrm{AdS}_5\times\mathrm{S}^5$

The map:
- $\sqrt{\lambda} \mapsto 4\pi h(\lambda)$
- Bethe mode number shifted by $1/2$
- $E_{\mathrm{AdS}_5} \mapsto 2E_{\mathrm{AdS}_4}$ (twice as many excitations)
- $S_{\mathrm{AdS}_5} \mapsto 2S_{\mathrm{AdS}_4}$ (BPS relation)

Bethe Ansatz vs. The Worldsheet

eternal problem: how to do reliable worldsheet perturbation theory and identify correctly the gauge theory and string theory parameters

eternal solution: Focus on states with large quantum numbers; worldsheet semiclassical expansion is reliable; identify the gauge theory operator by matching its charges; the charge and the "size" of the worldsheet are related

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◇ Two important solutions:

1) spinning folded string GKP; Frolov, Tseytlin

2) circular rotating string with 2 angular momenta Park, Tirziu, Tseytlin

- both exist in $AdS_3 \times S^1 \subset AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$
- both exhibit minimal structural changes compared to $AdS_5 \times S^5$
- main difference related to RR fields
- potentially expose subtle differences between the two models

The action: Bosonic part: sigma model based on the metrics

$$ds^2_{\mathrm{AdS}_4} = -\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\begin{aligned} ds^2_{\mathbb{CP}^3} &= d\zeta_1^2 + \sin^2\zeta_1\Big[d\zeta_2^2 + \cos^2\zeta_1\left(d\tau_1 + \sin^2\zeta_2\left(d\tau_2 + \sin^2\zeta_3 d\tau_3\right)\right)^2 \\ &+ \sin^2\zeta_2\left(d\zeta_3^2 + \cos^2\zeta_2\left(d\tau_2 + \sin^2\zeta_3 d\tau_3\right)^2 + \sin^2\zeta_3\cos^2\zeta_3 d\tau_3^2\right)\Big] \end{aligned}$$

- Coordinates iterativelly embedding $\mathbb{CP}^{n-1}$ into $\mathbb{CP}^n$ — Hoxha et al

- Radii: $R^2_{\mathbb{CP}^3} = 4R^2_{\mathrm{AdS}}$ $\quad R^2_{\mathrm{AdS}} = \dfrac{R^3}{4k_{\mathrm{CS}}} = \pi\sqrt{2\lambda} = \sqrt{\bar{\lambda}} \equiv$ string tension

Fermionic part: *complete* all-order GS action is not clear

V1. Use $\mathrm{AdS}_4 \times \mathbb{CP}^3 = SO(3,2)/SO(3,1) \times SU(4)/SU(3) \times U(1)$ and fit in a supergroup: $OSp(6|4)/SO(3,1) \times SU(4)/SU(3) \times U(1)$

Arutyunov, Frolov; Stefanski; Fre, Grassi

- only 24 fermions; partial κ-gauge-fixed; needs motion on $\mathbb{CP}^3$

V2. Double dimensional reduction from supermembrane in $\mathrm{AdS}_4 \times \mathrm{S}^7$

V3. Perturbative construction in number of fermions (need only θ^2)

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V1. GS on $OSp(6|4)/SO(3,1)\times SU(4)/SU(3)\times U(1)$

Arutyunov, Frolov; Stefanski; Fre, Grassi

- only 24 fermions; partial κ-gauge-fixed; needs motion on $\mathbb{CP}^3$
- Clasically integrable; classical transfer matrix
- Interesting open quantum question: conservation of higher charges is anomalous in sigma models on $\mathbb{CP}^n$ and cancels in ws susy situations; are GS fermions equally powerful?

- Assume all is well; discretize classical BE; conjecture all-order

Gromov, Vieira

Semiclassical expansion:

$$S = \frac{R^2_{\text{AdS}}}{2\pi}\int d\tau \int_0^{2\pi} d\sigma \sqrt{-g} g^{ab} \frac{1}{2}\partial_a X^M \partial_a X^N G_{MN}(X) \qquad R^2_{\text{AdS}} = \sqrt{\bar\lambda}$$

- $\bar\lambda = \lambda$ in $AdS_5 \times S^5$ while $\bar\lambda = 2\pi^2\lambda$ in $AdS_4 \times \mathbb{CP}^3$

Target space energy density

$$E = \sqrt{\bar\lambda}\,\mathcal{E}\left(\mathcal{S}_i, \mathcal{J}_i, \frac{1}{\sqrt{\bar\lambda}}\right) = \sqrt{\bar\lambda}\left[\mathcal{E}_0(\mathcal{S}_i, \mathcal{J}_i) + \frac{1}{\sqrt{\bar\lambda}}\mathcal{E}_1(\mathcal{S}_i, \mathcal{J}_i) + \dots\right]$$

Spin density R-charge density

$$S_i = \sqrt{\bar\lambda}\,\mathcal{S}_i \qquad J_i = \sqrt{\bar\lambda}\,\mathcal{J}_i$$

- Charges = identify the Cartan-s; phases of embedding coord's

Magnon dispersion relation at strong coupling: $\exists$ 8 bosonic exc.

- BMN limit using one of the Cartan isometries Nishioka, Takayanagi

$$\epsilon_{L,H} = \sqrt{n_{L,H} + 4\pi^2 h(\lambda)^2 \frac{k^2}{J^2}} \qquad h(\lambda) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(1)$$

- Bethe ansatz: leading correction to $h(\lambda)$ vanishes Shenderovich

- Tractable limit of the spinning folded string with finite charges:

$S \gg J \gg 1 \quad l = \frac{J}{\sqrt{\bar{\lambda}} \ln S}$ =fixed $\rightarrow$ homogeneous in w.s. coordinates

$$\bar{t} = \kappa\tau \qquad \bar{\rho} = \mu\sigma \qquad \bar{\phi} = \kappa\tau \qquad \bar{\varphi}_2 = \bar{\varphi}_3 = \frac{1}{2}\nu\tau \qquad \mu^2 = \kappa^2 - \nu^2$$

$\uparrow$ Virasoro constraints

$$(\mathcal{E}, \mathcal{S}, \mathcal{J}) = \int_0^{2\pi} d\sigma \, \frac{1}{2}(\kappa \cosh^2 \bar{\rho}, \kappa \sinh^2 \bar{\rho}, \nu)$$

$\mu = \frac{1}{\pi} \ln \mathcal{S} \quad \mu \gg 1 \qquad l = \frac{\nu}{\mu}$ can define $\mu\sigma$ as spatial ws coordinate

$\longrightarrow$ string length is effectively infinite

$\longrightarrow$ μ-dependence factorizes

- Leading order value of the space-time energy

$$E_0 - S \;=\; \sqrt{\bar{\lambda}} \ln S \sqrt{1 + l^2} = \sqrt{\bar{\lambda}}\, f_0(l) \ln S$$

- General behavior: $E - S = \sqrt{\bar{\lambda}}\, f(\bar{\lambda}, l) \ln S$

$\uparrow$ universal scaling function

Circular rotating string:

$$\bar{t}=\kappa\tau \qquad \bar{\rho}=\rho_* \qquad \bar{\theta}=\frac{\pi}{2} \qquad \bar{\phi}=\mathrm{w}\tau+k\sigma \qquad \bar{\varphi}_2=\bar{\varphi}_3=\frac{1}{2}(\omega\tau+m\sigma)$$

- Virasoro constraints and eq's of motion ($\mathrm{r}_0\equiv\cosh\rho_*$ and $\mathrm{r}_1\equiv\sinh\rho_*$)

$$\mathrm{w}^2-(\kappa^2+k^2)=0 \quad \mathrm{r}_1^2\mathrm{w}k+\omega m=0 \quad \mathrm{r}_0^2\kappa^2-\mathrm{r}_1^2(\mathrm{w}^2+k^2)-\omega^2-m^2=0$$

- Classical energy and charges

$$E_0=\sqrt{\bar{\lambda}}\,r_0^2\kappa \qquad S=\sqrt{\bar{\lambda}}\,\mathrm{r}_1^2\mathrm{w} \qquad J\equiv J_2=J_3=\sqrt{\bar{\lambda}}\,\omega$$

- Express E_0 in terms of charges and winding numbers k and m

in the scaling limit $S,J\to\infty$ with $u=S/J$-fixed

$$\begin{aligned}E_0 \;=\;& S+J+\frac{\bar{\lambda}}{2J}k^2u(1+u)-\frac{\bar{\lambda}^2}{8J^3}k^4u(1+u)(1+3u+u^2)\\ &+\frac{\bar{\lambda}^3}{16J^5}k^6u(1+u)(1+7u+13u^2+7u^3+u^4)+\mathcal{O}\left(\frac{1}{J^7}\right)\end{aligned}$$

◇ two possible relations between AdS_5 and AdS_4 results

1) $\bar{\lambda}_{\mathrm{AdS}_5}\mapsto\bar{\lambda}_{\mathrm{AdS}_4}$ 2) $E_{\mathrm{AdS}_5}\mapsto 2E_{\mathrm{AdS}_4}$, $J_{\mathrm{AdS}_5}\mapsto 2J_{\mathrm{AdS}_4}$, $\bar{\lambda}_{\mathrm{AdS}_5}\mapsto 4\bar{\lambda}_{\mathrm{AdS}_4}$

Quantum corrections:

V1. Hamiltonian formalism; works great with static gauge $t = \kappa\tau$

Frolov, Tseytlin

$$E = \frac{1}{\kappa}\langle\Psi|H|\Psi\rangle \quad \longrightarrow \quad E_1 = \frac{1}{\kappa}\langle\Psi|H_2|\Psi\rangle$$

fermion number $\searrow$ $\qquad$ $\uparrow$ Hamiltonian of quadratic fluctuations

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}(-)^{F_i}\omega_{n,i} \quad \longleftarrow \quad \text{fluctuation frequencies}$$

V2. Lagrangian formalism in conformal gauge

Frolov, Tirziu, Tseytlin

RR, Tseytlin

Large charges $\rightarrow$ the partition function localizes around a single critical point of the action; correction to energy from free energy while accounting for renormalization of the other charges

$$E_1 \propto \ln \operatorname{sdet} K \quad \mapsto \quad E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}(-)^{F_i}\omega_{n,i}$$

◇ carries over to higher loops

RR, Tseytlin

Quantum corrections:

- detailed knowledge of quadratic part of the action

  – from all-order action based on $OSp(6|4)/SU(3)\times U(1)\times SO(3,1)$

  Arutyunov, Frolov; Stefanski; Fre, Grassi

  used by Alday, Arutyunov, Bykov; Krishnan for SFS

  – General κ-symmetric form implying linearized sugra constraints

$$L_{2F} = i(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \not{e}_a D_b^{JK}\theta^K$$

↑ supercovariant derivative

$$\mathcal{D}_b = \partial_b + \frac{1}{4}\partial_b X^M \omega_M{}^{AB}\Gamma_{AB}$$

Hassan; Grana

$$\begin{aligned} D_b^{JK} &= \mathcal{D}_b\delta^{JK} - \frac{1}{8}\partial_b X^M E_M^A H_{ABC}\Gamma^{BC}(\sigma_3)^{JK} \\ &+ \frac{1}{8}e^{\phi}\left[F_{(0)}(\sigma_1)^{JK} + \not{F}_{(2)}(i\sigma_2)^{JK} + \not{F}_{(4)}(\sigma_1)^{JK}\right]\not{e}_b \end{aligned}$$

- In special Lorentz frame $\mathbb{CP}^3$ spin connection is not important
- After appropriate rotations projector is exposed; fix κ-symmetry

e.g. SFS: $L = i\bar{\Psi}(\eta^{ab} - \epsilon^{ab}\Gamma_{11})(\tau_a\partial_b + \tau_a\widehat{M}\tau_b)\Psi$, $\Psi = S^{-1}\theta$, $S = \exp(\kappa/2\sigma\Gamma_{a3})$

Spectrum of quadratic fluctuations; spinning folded string

($\Phi = \bar{\Phi} + \epsilon\tilde{\Phi}$ and rotation on $\tilde{\Phi}$) McLoughlin, RR

- Bosons:
  - two massless modes (one in AdS_4; one in $\mathbb{CP}^3$); canceled by ghosts
  - three modes from AdS_4

$$\omega_{\pm}(n) = \sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2\nu^2}} \qquad \omega_T(n) = \sqrt{n^2 + 2\kappa^2 - \nu^2}$$

  - one+four modes from $\mathbb{CP}^3$ (reflects breaking $SO(6) \to SO(4)$)

$$\omega_H(n) = \sqrt{n^2 + \nu^2} \qquad 4 \text{ of } \omega_L(n) = \sqrt{n^2 + \frac{1}{4}\nu^2}$$

- Fermions: (reflects breaking $SO(6) \to SO(4)$)

$$\omega_{\pm 12}(n) = \pm\frac{\nu}{2} + \sqrt{n^2 + \kappa^2}$$

$$\omega_{\pm 34}(n) = \frac{1}{\sqrt{2}}\sqrt{n^2 + 2\kappa^2 \pm \sqrt{\kappa^4 + 4n^2\nu^2}}$$

$$e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4\omega_L - \sum_{i=1}^{4}(\omega_{+i} + \omega_{-i}) \qquad E_1 = \sum_n e(n)$$

◇ Ws of infinite length ⇒ sum ↦ integal

$$e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4\omega_L - \sum_{i=1}^{4} (\omega_{+i} + \omega_{-i}) \qquad E_1 = \int_0^\infty dp e(\kappa p)$$

a) $(S, J = 0)$: $E_1 = -\frac{5 \ln 2}{2\pi} \ln S + \mathcal{O}\left(\ln^0 S\right)$

b) $(S, J \neq 0)$ $\left(u = \frac{l}{\sqrt{1+l^2}} \quad l = \frac{J}{\sqrt{\bar{\lambda}_{\mathrm{AdS}_4}} \ln S}\right)$

$$E_1 = \frac{\nu}{2u}\Big[-(1-u^2) + \sqrt{1-u^2} - 2u^2 \ln u - (2-u^2) \ln\left(\sqrt{2-u^2}(1+\sqrt{1-u^2})\right) - 2(1-u^2)\ln 2 \Big]$$

• contrast with $\mathrm{AdS}_5 \times \mathrm{S}^5$ energy shift

a) $(S, J = 0)$: $E_1 = -\frac{3 \ln 2}{2\pi} \ln S + \mathcal{O}\left(\ln^0 S\right)$

b) $(S, J \neq 0)$ $\left(u = \frac{l}{\sqrt{1+l^2}} \quad l = \frac{J}{\sqrt{\lambda_{\mathrm{AdS}_5}} \ln S}\right)$

$$E_1 = \frac{\nu}{2u}\Big[-(1-u^2) + \sqrt{1-u^2} - 2u^2 \ln u - (2-u^2) \ln\left(\sqrt{2-u^2}(1+\sqrt{1-u^2})\right) \Big]$$

Spectrum of quadratic fluctuations; circular rotating string

($\Phi = \bar{\Phi} + \epsilon\tilde{\Phi}$ and rotation on $\tilde{\Phi}$) McLoughlin, RR, Tseytlin

- Bosons:
  - two massless modes (one in AdS_4; one in $\mathbb{CP}^3$); canceled by ghosts
  - three modes from AdS_4: $\omega_T(n) = \sqrt{p_1^2 + \kappa^2}$ & two solutions of

$$\frac{1}{4}(\omega(n)^2 - n^2)^2 + r_1^2\kappa^2\omega(n)^2 - \left(1 + r_1^2\right)\left(\sqrt{\kappa^2 + k^2}\omega(n) - kn\right)^2 = 0$$

  - one+four modes from $\mathbb{CP}^3$ (reflects breaking $SO(6) \to SO(6)$)

$$\omega_H(n) = \sqrt{n^2 + (\omega^2 - m^2)} \qquad 4 \text{ of } \omega_L(n) = \sqrt{n^2 + \frac{1}{4}(\omega^2 - m^2)}$$

- Fermions: (reflects breaking $SO(6) \to SO(6)$)

$$\omega_{\pm 12}(n) = \pm\frac{r_0^2 k\kappa m}{2(m^2 + r_1^2 k^2)} + \sqrt{(p_1 \pm b)^2 + (\omega^2 + k^2 r_1^2)} \ ; \ b = -\frac{\kappa m}{w}\frac{w^2 - \omega^2}{2(m^2 + r_1^2 k^2)}$$

$$(\omega(n)^2 - n^2)^2 + r_1^2\kappa^2\omega(n)^2 - \left(1 + r_1^2\right)\left(\sqrt{\kappa^2 + k^2}\omega(n) - kn\right)^2 = 0$$

$$e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4\omega_L - \sum_{i=1}^{4}(\omega_{+i} + \omega_{-i}) \qquad E_1 = \frac{1}{2\kappa}\sum_n e(n)$$

- Scaling limit: $S, J \to \infty$ with fixed $u = S/J$; expand in $1/J$
  - features similar to $AdS_5 \times S^5$ calculation: sum at finite J and S is convergent but some terms in the expansion lead to divergent contributions

Beisert, Tseytlin for $AdS_5 \times S^5$

- e.g. leading term in the scaling limit ($J = \sqrt{\lambda}\omega, \ n = \omega x$)

$$e^{\text{sum}}(n) = \frac{1}{2\omega}\left[n\left(3n - 4\sqrt{n^2 + k^2 u(1+u)} + \sqrt{n^2 + 4k^2 u(1+u)}\right) - k^2(1+u)(1+3u)\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)$$

$$e^{\text{int}}(x) = \frac{k^2(1+u)}{2\omega}\left[\frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2\frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}}\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)$$

- $e^{\text{int}}(0) = e^{\text{sum}}(0) \to$ ignore last term in $e^{\text{sum}}(n)$ and replace its contribution with the integral of $e^{\text{int}}(x)$; resummation of divergences

- Direct numerical evaluation confirms this interpretation

- Scaling limit: $S, J \to \infty$ with fixed $u = S/J$; expand in $1/J$
  - features similar to AdS$_5\times$S^5 calculation: sum at finite J and S is convergent but some terms in the expansion lead to divergent contributions
    Beisert, Tseytlin for AdS$_5\times$S^5

- e.g. leading term in the scaling limit ($J = \sqrt{\lambda}\omega,\ n = \omega x$)

$$e^{\text{sum}}(n) = \frac{1}{2\omega}\left[n\left(3n - 4\sqrt{n^2 + k^2 u(1+u)} + \sqrt{n^2 + 4k^2 u(1+u)}\right)\right.$$
$$\left. - k^2(1+u)(1+3u)\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)$$

$$e^{\text{int}}(x) = \frac{k^2(1+u)}{2\omega}\left[\frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2\frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}}\right] + \mathcal{O}\left(\frac{1}{\omega^3}\right)$$

- $\sum_n \mapsto \omega\int_{-\infty}^{+\infty} dx \Rightarrow \begin{cases} e^{\text{sum}} \\ e^{\text{int}} \end{cases}$ are expansions in $\begin{cases} 1/J^{\text{even}} \\ 1/J^{\text{odd}} \end{cases}$

$\to$ analyze separately

$$
\begin{aligned}
E_1^{\text{odd}} &= \frac{\omega}{2\kappa}\int_{-\infty}^{\infty} dx\; e_{\text{reg}}^{\text{int}}(x) \\
&= -\frac{\bar{\lambda}^{1/2}k^2}{J}\ \ln 2\ u(1+u) + \frac{\bar{\lambda}^{3/2}k^4}{2J^3}\ \ln 2\ u(1+u)(1+3u+u^2) \\
&\quad -\frac{\bar{\lambda}^{5/2}k^6}{8J^5}u(1+u)\Big[3(1+7u+13u^2+7u^3+u^4)\ln 2\Big] \\
&\quad +\frac{\bar{\lambda}^{5/2}k^6}{6J^5}u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)
\end{aligned}
$$

- combine with leading order terms

$$
\begin{aligned}
E_0 + E_1^{\text{odd}} &= S + J + \frac{\bar{h}^2(\bar{\lambda})k^2}{2J}u(1+u) - \frac{\bar{h}^4(\bar{\lambda})k^4}{8J^3}u(1+u)(1+3u+u^2) \\
&\quad + \frac{\bar{h}^6(\bar{\lambda})k^6}{16J^5}u(1+u)(1+7u+13u^2+7u^3+u^4) \\
&\quad + \frac{\bar{h}^5(\bar{\lambda})k^6}{6J^5}u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)
\end{aligned}
$$

- introduce $\bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right)$; to this order $\bar{h}(\bar{\lambda})^n$ contributes the first two terms in its expansion

- combine with leading order terms $\bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right)$

$$(E_0 + E_1^{\text{odd}})_{\text{AdS}_4\times\mathbb{CP}^3} = S + J + \frac{\bar{h}^2(\bar{\lambda})k^2}{2J}u(1+u) - \frac{\bar{h}^4(\bar{\lambda})k^4}{8J^3}u(1+u)(1+3u+u^2) + \frac{\bar{h}^6(\bar{\lambda})k^6}{16J^5}u(1+u)(1+7u+13u^2+7u^3+u^4) + \frac{\bar{h}^5(\bar{\lambda})k^6}{6J^5}u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)$$

$$(E_0 + E_1^{\text{odd}})_{\text{AdS}_5\times\text{S}^5} = J + S + \frac{\lambda_{\text{AdS}_5}k^2}{2J}u(1+u) - \frac{\lambda^2_{\text{AdS}_5}k^4}{8J^3}u(1+u)(1+3u+u^2) + \frac{\lambda^3_{\text{AdS}_5}k^6}{16J^5}u(1+u)(1+7u+13u^2+7u^3+u^4) + \frac{\lambda^{5/2}_{\text{AdS}_5}k^6}{3J^5}u^3(1+u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)$$

◇ The map: $E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}, \; J_{\text{AdS}_5} \mapsto 2J_{\text{AdS}_4}, \; \bar{\lambda}_{\text{AdS}_5} \mapsto 4\bar{h}^2(\bar{\lambda}_{\text{AdS}_4})$

after all parameters of the solution are expressed in terms of charges!

$$(\bar{E}_1^{\text{even}})_{\text{AdS}_4\times\mathbb{CP}^3} = \frac{1}{\kappa}\sum_{n=1}^{\infty} e_{\text{reg}}^{\text{sum}}(n)$$

$$= -\frac{\bar{\lambda}k^4(1+u)^2u^2}{2^3J^2}\left(6\zeta(2) - 15k^2u(1+u)\zeta(4) + \frac{315}{8}k^4u^2(1+u)^2\zeta(6) + \dots\right)$$
$$+\frac{\bar{\lambda}^2k^6(1+u)^2u^2}{2^6J^4}\Big(24(1+2u-u^2)\zeta(2) + 15k^2u^2(1+u)(5+13u)\zeta(4)$$
$$-\frac{63}{2}k^4u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \dots\Big) + \mathcal{O}\left(\frac{1}{J^6}\right)$$

$$(\bar{E}_1^{\text{even}})_{\text{AdS}_5\times\text{S}^5} = \frac{1}{\kappa}\sum_{n=1}^{\infty} e_{\text{reg, AdS}_5\times\text{S}^5}^{\text{sum}}(n)$$

$$= -\frac{\lambda k^4(1+u)^2u^2}{2^2J^2}\left(4\zeta(2) - 8k^2u(1+u)\zeta(4) + 20k^4u^2(1+u)^2\zeta(6) + \dots\right)$$
$$+\frac{\lambda^2k^4(1+u)^2u^2}{2^5J^4}\Big(16k^2(1+2u-u^2)\zeta(2) + 8k^4u^2(1+u)(5+13u)\zeta(4)$$
$$-16k^6u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \dots\Big) + \mathcal{O}\left(\frac{1}{J^6}\right)$$

- Besides $E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}$, $J_{\text{AdS}_5} \mapsto 2J_{\text{AdS}_4}$, $\bar{\lambda}_{\text{AdS}_5} \mapsto 4\bar{h}^2(\bar{\lambda}_{\text{AdS}_4})$, mapping the two expressions into each other requires a re-identification of zeta-constants; physically unjustified $\mapsto$ differs from proposed BA

Conclusions

- The natural worldsheet and (built in) Bethe Ansatz regularization schemes are not necessarily the same

- Magnon dispersion relation receives (in conformal gauge) scheme-dependent corrections

- Remains an open question whether all quantities depend only on $\bar{h}(\bar{\lambda})$; if so, choose some anomalous dimension as physical coupling

- conjectured all-loop Bethe Ansatz reproduces (the continuous) part of the worldsheet results; finite size effects need more analysis
  - giant magnon finite size effects seem to work out fine, however

Grignani, Harmark, Orselli, Semenoff; Bombardelli, Fioravanti; Lukowski, Sax; Ahn, Bozhilov;...

- Possible origin of differences
  - misidentification of sectors/excitations
  - misidentification of S-matrix, especially S_{AB}
  - breakdown of integrability at the quantum level