

Some Basic Numerical Uncertainty Analysis

General Description

Every measurement you ever make will have an uncertainty associated with it. When you use that measurement to determine another quantity, the uncertainty propagates through your calculations. The uncertainties from every measurement you make have an effect on the uncertainty on your final result. A simple method of quantifying this effect is to look at the upper and lower limits of the original measurement. Here is a general description of how to do this.

Say you measure some value, x , with some uncertainty, σ_x , so that your full measurement is given as $x \pm \sigma_x$. You wish to find a new value that depends on x , and we'll denote this value as $f(x)$. Finding $f(x)$ is simple; you just plug in x . To find the uncertainty of $f(x)$ due to σ_x , we must look at the upper and lower bounds of x itself. We know the true value for x lies somewhere between $x + \sigma_x$ and $x - \sigma_x$. Therefore, we can say that the true value for $f(x)$ lies somewhere between $f(x + \sigma_x)$ and $f(x - \sigma_x)$. Thus, the overall uncertainty for $f(x)$ is given by

$$\sigma_{f(x)} = \frac{f(x + \sigma_x) - f(x - \sigma_x)}{2}$$

Now, your final result is given as $f(x) \pm \sigma_{f(x)}$. So that's it, but if you're anything like me, you'll want to see an example.

Example

Let's say we're timing how long it takes a ball to hit the ground after we drop it off a tall building. The point of this is to determine the height of the building, and we can do this using the equation

$$h = \frac{1}{2}gt^2$$

In this equation, h is the height of the building in meters, g is acceleration of an object due to gravity (9.8 m/s²), and t is the time it takes for the object to hit the ground, measured in seconds.

Our stopwatch measures in increments of 0.1 seconds. If you remember, this means that we have a 0.05 second systematic uncertainty in all of our time measurements, simply caused by the limitations of our measuring device (I should note that there are other uncertainties due to things like air resistance and the reaction time of the person

holding the stopwatch, but for the sake of simplicity, we'll focus only on the systematic uncertainty of the stopwatch itself). We drop our ball, and we measure 4.3 seconds for it to hit the ground, so our full measurement is given as 4.3 ± 0.05 seconds. Plugging this in to our formula for h , we find

$$h = \frac{1}{2}(9.8)(4.3)^2 = 90.6 \text{ meters}$$

Now, let's find the uncertainty in h . We know our measured time falls between 4.25 and 4.35 seconds, so h lies between

$$\frac{1}{2}(9.8)(4.25)^2 = 88.5 \text{ meters}$$

and

$$\frac{1}{2}(9.8)(4.35)^2 = 92.7 \text{ meters}$$

Therefore, our uncertainty in h is given by

$$\sigma_h = \frac{92.7 - 88.5}{2} = 2.1 \text{ meters}$$

We report our final height as $h = 90.6 \pm 2.1$ meters.

It's tempting to want to plug the time uncertainty, σ_t , straight into the equation for h , but you shouldn't do this. If you do, you get an uncertainty of

$$\sigma_h = \frac{1}{2}(9.8)(0.05)^2 = 0.01 \text{ meters} = 1 \text{ cm}$$

It would be great if our uncertainty was that small, but this is a drastic underestimation of the true value. So be careful!