Algebraic Approach to the Scattering Matrix

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A purely algebraic procedure is presented for calculating recursion relations for S matrices belonging to problems associated with the group SU(1,1). The procedure involves supplementing a recently introduced group-theoretic approach to scattering by an algebraic framework that characterizes asymptotic behavior. Our formulation makes use of the Euclidean group, which contains the symmetry transformations for a free particle. The Coulomb and Pöschl-Teller potentials are discussed to illustrate our methods.

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Dynamical groups have proved to be useful in describing bound-state spectra in nuclei[1] and molecules.[2] Recently[3,4] the group-theoretic approach has been extended to the continuum. The methods were illustrated for one-dimensional Pöschl-Teller and Morse potentials, whose scattering eigenstates were shown to form a basis for certain representations of the noncompact group SU(1,1).[5] Two types of dynamical SU(1,1)'s were considered. In the first, called the scattering-state group, a given representation contained eigenstates with varying energies corresponding to a fixed potential strength; in the second, known as the potential group, the basis consisted of states with a fixed energy but corresponding to different values of the potential strength.

Despite the algebraic classification of the continuous spectrum, it was not known yet how to calculate the S matrix in a purely algebraic way. The reason was that the S matrix is defined through the asymptotic behavior of the scattering states, and in a framework in which a state is described simply as an abstract vector in some group representation space, it was not clear how to incorporate the notion of an asymptotic limit. In an earlier treatment[6] of the Coulomb problem, the only known example in which an S matrix was calculated algebraically, the difficulty was avoided, but in a way that could not be generalized to other problems. More recently,[7] a dynamical SU(1,1) potential group was used to derive the S matrix for the Pöschl-Teller potential. However, the method employed was not completely algebraic because in exploiting the asymptotic behavior of the scattering states the algebraic language was abandoned and use was made of an explicit realization of SU(1,1).

The purpose of this Letter is to construct an algebraic framework to characterize asymptotic behavior, so that calculations like those in Ref. 7 may be recast in purely algebraic form and then generalized to models like those in Refs. 1 and 2 in which only a dynamical group Hamiltonian is specified and no differential Schrödinger equation is available. Here we shall only consider problems in which the associated group is either a symmetry group or a dynamical potential group,[4] both of which have the property that they leave the energy invariant. Our formulation of the asymptotic limit will then make use of the Euclidean group. This is quite natural since scattering states behave asymptotically like free waves and the Euclidean group contains the symmetry operations (translations and rotations) that leave the free-particle energy invariant. The interplay between the dynamical group of the problem and the Euclidean group will provide us with the machinery to calculate recursion relations for S matrices in a completely algebraic way. This procedure, when generalized, may be of use in an algebraic treatment of atomic and nuclear collisions similar to that which exists for bound states.[1,2]

We first illustrate our approach with two examples discussed in parallel: two-dimensional Coulomb scattering and one-dimensional Pöschl-Teller[8] potential. The Coulomb Hamiltonian is $H_C=\vec{P}^2-\alpha/r$ where $\vec{P}$ is the two-dimensional momentum operator and r and ϕ are polar coordinates in the x_1-x_2 plane. There is an SO(2,1) [$\sim$SU(1,1)] symmetry group[9] generated within a subspace of constant energy, k^2, by the angular momentum (in two dimensions) $L_z=-i\,\partial/\partial\phi$ and the Lenz vector[10] $2kL_\pm$ where $L_\pm=[-\alpha\hat{r}_\pm \mp i(P_\pm L_z+L_zP_\pm)]/2k$. Here $\hat{r}_\pm=e^{\pm i\phi}$, and $P_\pm=P_1\pm iP_2$ are the spherical components of the radial unit vector and the momentum. These generators satisfy the SU(1,1) commutation relations

$$[L_z,L_\pm]=\pm L_\pm,\quad [L_+,L_-]=-2L_z. \tag{1}$$

The Coulomb Hamiltonian is related to the Casimir invariant $C=L_z^2-(L_+L_-+L_-L_+)/2$ of (1) by $H_C=-(\alpha^2/4)(C+\frac{1}{4})^{-1}$, so that the scattering eigenstates with energy k^2 and angular momentum m form a basis $|j,m\rangle$ for an SU(1,1) representation in which

$$C|j,m\rangle=j(j+1)|j,m\rangle,\quad L_z|j,m\rangle=m|j,m\rangle. \tag{2}$$

Here $j=-\frac{1}{2}+i(\alpha/2k)$ so that the above representation belongs to the principal continuous series and contains $m=0, \pm 1, \pm 2, \ldots$. After a similarity transformation $\sqrt{r}$ we find

$$|j,m\rangle = e^{im\phi}\Psi_{jm}(r), \tag{3}$$

where Ψ_{jm} satisfies the radial equation

$$\left[-\frac{d^2}{dr^2}-\frac{\alpha}{r}+\frac{m^2-\frac{1}{4}}{r^2}\right]\Psi_{jm}=k^2\Psi_{jm}, \quad r\geq 0. \tag{4}$$

The one-dimensional Pöschl-Teller problem can be cast in a similar algebraic form provided that we imbed the one dimension in a two-dimensional space. The resulting SU(1,1) is a dynamical potential group[11] rather than a symmetry group and is obtained through the assignments[7] $L_z=-i\partial/\partial\phi$,

$$L_\pm = e^{\pm i\phi}[-\partial/\partial r+\tanh r(\pm\tfrac{1}{2}-i\,\partial/\partial\phi)].$$

The Casimir invariant is then

$$C=\partial^2/\partial r^2-(\partial^2/\partial\phi^2+\tfrac{1}{4})/\cosh^2 r-\tfrac{1}{4}.$$

Defining the states $|j,m\rangle$ by the same set of algebraic equations (2) we find that equation (3) still holds but now Ψ_{jm} satisfies the Pöschl-Teller equation

$$\left[-\frac{d^2}{dr^2}-\frac{m^2-\frac{1}{4}}{\cosh^2 r}\right]\Psi_{jm}=k^2\Psi_{jm}, \quad -\infty<r<\infty, \tag{5}$$

where $j=-\frac{1}{2}+ik$. We can therefore write the Pöschl-Teller Hamiltonian in the form $H_{\rm PT}=-(C+\frac{1}{4})$. Note that here the states $|j,m\rangle$ belonging to a representation j have *fixed* energy k^2 but correspond to different potential strengths $m^2-\frac{1}{4}$. Thus, the potential group of this problem plays a similar role to that of the Coulomb symmetry group. Both leave the energy invariant, and change a parameter m that measures the strength of a potential (the centrifugal barrier in the Coulomb case).

To obtain the S matrix in either example, it is necessary to consider the asymptotic limit[12] $r\to\infty$. We define the asymptotic scattering states by $|j,m\rangle^\infty \equiv \lim_{r\to\infty}|j,m\rangle$ and the asymptotic generators $L_\pm^\infty, L_z^\infty$ are similarly obtained from $L_\pm, L_z$. Since the algebraic properties of the generators and the states are preserved in the asymptotic limit, it is clear that the asymptotic generators $L_\pm^\infty, L_z^\infty$ still form an SU(1,1) algebra

$$L_\pm^\infty|j,m\rangle^\infty=[m-j)(m+j+1)]^{1/2}|j,m\pm 1\rangle^\infty, \qquad L_z^\infty|j,m\rangle = m|j,m\rangle. \tag{6}$$

In both examples (provided the Pöschl-Teller problem corresponds to a wave from the right) we have in the r,ϕ realization

$$|j,m\rangle^\infty = A_m e^{im\phi}e^{-ikr}+B_m e^{im\phi}e^{ikr}, \tag{7}$$

where A_m,B_m are k-dependent constants and the appropriate relation between j and k is to be used in each example.

In Ref. 7 recursion relations for A_m,B_m were obtained in the Pöschl-Teller problem by operating with the raising operator L_+^∞ on both sides of (7), but on the right-hand side (rhs) the explicit realization had to be used. What we need now is a way to formulate the procedure algebraically, that is without reference to the variables r,ϕ. We do this by observing that the asymptotic wave functions correspond in both examples to a free particle in two dimensions. As explained before, we can therefore use the Euclidean group in two dimensions,[13] E(2), to characterize these asymptotic states. The group E(2) has three generators: the linear momenta P_1,P_2 and the angular momentum L_z. They satisfy the commutation relations

$$[L_z,P_\pm]=\pm P_\pm, \quad [P_+,P_-]=0, \tag{8}$$

where $P_\pm=P_1\pm iP_2$. The asymptotic form of these generators in polar coordinates is given by

$$P_\pm^\infty=\lim_{r\to\infty}P_\pm=-i\hat{r}_\pm\frac{\partial}{\partial r}, \qquad L_z^\infty=L_z=-i\frac{\partial}{\partial\phi}. \tag{9}$$

where $P_\pm^\infty, L_z$ clearly still obey the E(2) commutation relations (8).

Now we observe that each of the waves on the rhs of (7) forms (for a given k) a representation for the asymptotic E(2) algebra. Introducing the kets

$$|\pm k,m\rangle = e^{im\phi}e^{\pm ikr} \tag{10}$$

to denote outgoing (k) and incoming $(-k)$ circular waves of energy k^2 and angular momentum m, we find

$$\begin{aligned} P_\pm^\infty|k,m\rangle &= k|k,m\pm 1\rangle, \\ L_z|k,m\rangle &= m|k,m\rangle, \\ P^2|k,m\rangle &= k^2|k,m\rangle; \\ P_\pm^\infty|-k,m\rangle &= -k|-k,m\pm 1\rangle, \\ L_z|-k,m\rangle &= m|-k,m\rangle, \\ P^2|-k,m\rangle &= k^2|-k,m\rangle, \end{aligned} \tag{11}$$

where $P^2=P_\mp^\infty P_\pm^\infty$ is the E(2) Casimir invariant.

The two irreducible representations[13] (11) of E(2), corresponding to all outgoing and all incoming waves of energy k^2, are labeled by k and $-k$, respectively. Using definition (10) we can rewrite Eq. (7) in the form

$$|j,m\rangle^{\infty} = A_m|-k,m\rangle + B_m|k,m\rangle. \quad (12)$$

We now view Eqs. (11) and (12) as abstract algebraic relations independent of their specific realizations (9) and (10). The states $|\pm k,m\rangle$ are defined simply as eigenstates of the E(2) operators P^2 and L_z and can be divorced from their coordinate realization (10).

We next need to represent within this algebraic framework the action of $L_{\mp}^{\infty}$ on the states $|\pm k,m\rangle$. Inspecting our examples we observe that

$$L_{\mp}^{\infty} = \begin{cases} -i\{[-\frac{1}{2} - i\alpha/2(\pm k)]P_{\mp}^{\infty} + L_z P_{\mp}^{\infty}\}/k & \text{Coulomb} \\ \{[-\frac{1}{2} - i(\pm k)]P_{\mp}^{\infty} + L_z P_{\mp}^{\infty}\}/k & \text{Pöschl-Teller} \end{cases} \quad (13)$$

where the $\pm k$ refer to the E(2) representation in which the operators act. We notice that in both cases $L_{\mp}^{\infty}$ is, for a given k, a function of the Euclidean generators $P_{\mp}^{\infty}, L_z$. Now we argue that this is a property of any SU(1,1) problem in which the asymptotic states are labeled by E(2). Since SU(1,1) leaves the energy k^2 invariant, so does its asymptotic version and $L_{\mp}^{\infty}$ must commute with P^2, the E(2) Casimir invariant. Since E(2) is the "maximal" symmetry group of the free particle, no other transformations (save discrete ones) preserving P^2 exist so that L_+ must be made up of E(2) operators. Once we have a relation of the type (13) we can get unique recursion relations for A_m and B_m by applying $L_{\mp}^{\infty}$ to Eq. (12), and using Eqs. (6) and (11). In our examples, where $L_{\mp}^{\infty}$ is given by (13), the reflection amplitude $R_m = B_m/A_m$ satisfies

$$R_{m+1}(k) = e^{i\delta}\frac{m+\frac{1}{2}-if(k)}{m+\frac{1}{2}+if(k)}R_m(k). \quad (14)$$

where $f(k) = \alpha/2k$, $\delta = \pi$ for the Coulomb problem and where $f(k) = k$, $\delta = 0$ for the Pöschl-Teller example. Solving (14) yields

$$R_m(k) = e^{im\delta}\frac{\Gamma(m+\frac{1}{2}-if(k))}{\Gamma(m+\frac{1}{2}+if(k))}\Delta(k), \quad (15)$$

where $\Delta(k)$ is an m-independent overall factor determined by $R_0(k)$.

As it stands, Eq. (15) holds only for integer values of m (for the Coulomb problem, in which m is the angular momentum, these are the only values of physical interest). The recursion relation (14), however, holds for any m; this can be readily shown by repeating our algebraic procedures for SU(1,1) representations in which $m = m_0, m_0 \pm 1, m_0 \pm 2, \ldots$ where $0 \leq m_0 < 1$. Thus, our methods serve to connect algebraically all amplitudes with m values differing by integers. In fact, relation (15) will also hold for m values starting for noninteger m_0, provided that $\Delta(k)$ is now determined by $R_{m_0}(k)$. It may happen that this factor $\Delta(k)$ is either physically unimportant or else is known. In the Coulomb problem, $\Delta(k)$ does not contribute to the differential cross section while in the Pöschl-Teller problem, in the special case $m_0 = \frac{1}{2}$, $R_{1/2}(k)$ is manifestly 0 implying that the potential is reflectionless for all half-integer m.

For the sake of clarity we have deferred until now discussion of an important connection between E(2) and SU(1,1). The Euclidean representations (11) which appear on the rhs of (12) can be directly constructed from the SU(1,1) representations (6) which are induced by the scattering states. This is accomplished by a limiting process known as contraction.[14] Defining $P_{\pm}^{\epsilon} = \epsilon L_{\pm}$ we see that in the limit $\epsilon \to 0$, $P_{\mp}^{\epsilon}, L_z$ satsify the E(2) commutation relations (8). The suitable limiting process for the representations is easily found by inspecting the Casimir invariants. In the limit $\epsilon \to 0$, the SU(1,1) representations with $j = -\frac{1}{2} + i\beta$ reproduce the Euclidean representations $\pm k$ (11) if $\beta \to \infty$ in such a way that $\epsilon\beta = \pm k$ is kept constant. For more details about contractions see Ref. 14.

As we have stressed, the real value of the method presented in this paper is that it provides a purely algebraic procedure for obtaining S-matrix recursion relations that may be generalized to problems in which the Hamiltonian is expressed in terms of generators of a group rather than as a differential operator. We now outline the procedure when the Hamiltonian is given as an arbitrary function of the Casimir operator of SU(1,1), which we interpret as a symmetry or potential group.

(i) Identify the SU(1,1) representation associated with each energy k^2. A general "scattering" representation is of the form $j = -\frac{1}{2} + if(k)$. Since $C = -\frac{1}{4} - f^2(k)$ and $H = k^2$, it is clear that the function $f(k)$ is determined by the relation connecting H and C.

(ii) Construct the incoming and outgoing asymptotic representations. In our case these are simply the E(2) representations (11). Note, however, that for more complicated groups the identification of these representations may be harder and a contraction process like the one described above for SU(1,1) can be useful.

(iii) Choose an allowed form for $L_\mp^\infty$ in terms of the E(2) generators. This form is restricted since the asymptotic generators $L_\mp^\infty, L_z$ must obey SU(1,1) commutation rules and reproduce the Casimir eigenvalue given in (i). In fact, the most general $L_\mp^\infty$ satisfying these requirements and containing terms up to second order in the E(2) generators is

$$L_\mp^\infty = \{\exp[i\gamma_\pm(k)]/k\}[-\tfrac{1}{2}+if(k)]P_\mp^\infty + L_z P_z^\infty\}, \tag{16}$$

where again the $\pm k$ refer to the $\pm k$ of the E(2) representations. The $\gamma_\pm(k)$ are arbitrary real functions and $f(k)$ is the same as in (i).

(iv) Apply $L_\mp^\infty$ to Eq. (12) to obtain recursion relations for the S matrix.

Step (iii) is clearly the crucial one. Any form chosen in (iii) will determine a unique recursion relation for the S matrix at a given energy. The reflection amplitude which corresponds to the general form (16) of $L_\mp^\infty$ is

$$R_m(k) = \exp\{i[\gamma_+(k)-\gamma_-(k)]m\}\frac{\Gamma(m+\frac{1}{2}-if(k))}{\Gamma(m+\frac{1}{2}+if(k))}\Delta(k). \tag{17}$$

The remarks following Eq. (15) hold here as well. Also, when the problem is invariant under $m \to -m$, the quantity $\gamma_+(k)-\gamma_-(k)$ must be either 0 or π (for m integer).

Another potential which is known[4] to be associated with an SU(1,1) potential group is the one-dimensional Morse oscillator and it must therefore be a special case of the above. This can indeed by shown[15] by using a different realization from that of Ref. 4.

It should be possible to generalize the method presented here to other groups and to higher-dimensional problems. An application of the method to the three-dimensional Coulomb problem where the relevant groups are O(3,1) and E(3) will be presented elsewhere.[15] We view our results here as an important extension of the work presented in a previous Letter[3] and are hopeful that we are closer to an algebraic description of atomic and nuclear collisions similar to that developed for bound-state spectra.[1,2]

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[1]A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976), and **111**, 201 (1978), and **123**, 468 (1979).

[2]F. Iachello, Chem. Phys. Lett. **78**, 581 (1981); F. Iachello and R. D. Levine, J. Chem. Phys. **77**, 3046 (1982).

[3]Y. Alhassid, F. Gürsey, and F. Iachello, Phys. Rev. Lett. **50**, 873 (1983).

[4]Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **148**, 346 (1983), and Chem. Phys. Lett. **99**, 27 (1983).

[5]B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).

[6]D. Zwanzinger, J. Math. Phys. **8**, 1858 (1967).

[7]A. Frank and K. B. Wolf, Phys. Rev. Lett. **52**, 1737 (1984).

[8]G. Pöschl and E. Teller, Z. Phys. **83**, 143 (1933).

[9]For a review, see M. J. Engelfield, *Group Theory and the Coulomb Problem* (Wiley, New York, 1972).

[10]Note that $L_\pm$ are not components of the angular momentum. The latter has only one component L_z in two dimensions.

[11]The potential group for a more general Pöschl-Teller potential was introduced by F. Gürsey, in *Group Theoretical Methods in Physics,* Lecture Notes in Physics No. 180 (Springer-Verlag, Heidelberg, 1983); Y. Alhassid, in *Boson in Nuclei* (World Scientific, Singapore, 1983).

[12]In the Pöschl-Teller case we also need to consider the limit $r \to \infty$. This limit can be studied in a similar way.

[13]W. Miller, Jr., Mem. Am. Math. Soc. **50**, (1964).

[14]E. İnönü, in *Group Theoretical Concepts and Methods in Elementary Particle Physics,* edited by F. Gürsey (Gordon and Breach, New York, 1964), pp. 391–402.

[15]Y. Alhassid and J. Wu, to be published.