Thank you for the invitation.

BINARY TETRAHEDRAL GROUP AND PREDICTIONS FOR PMNS AND KM ANGLES

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Paul H Frampton

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OUTLINE

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SUMMARY.

References:

(1) P.H.F. and S. Matsuzaki.

`arXiv: 0806.4592 [hep-ph]`

(2) P.H.F., T.W. Kephart and S. Matsuzaki.

Phys. Rev. **D 78,** 073004 (2008).
`arXiv:0807.4713 [hep-ph]`

(3) P.H.F. and S. Matsuzaki.

`arXiv: 0810.1029 [hep-ph]`

(4) D.A. Eby, P.H.F. and S. Matsuzaki.

`arXiv: 0810.4899 [hep-ph]`

1. Introduction on renormalizability

In particle theory phenomenology, model building fashions vary with time and because the present lack of data (soon to be compensated by the Large Hadron Collider) does not allow discrimination between models some fashions develop a life of their own. In the present Letter we take the apparently retrogressive step of imposing the requirement of renormalizability, as holds for quantum electrodynamics (QED), quantum chromodynamics (QCD) and the standard electroweak model, to show that non-abelian flavor symmetry becomes then much more restrictive and predictive. In a specific model we show that a normal neutrino mass hierarchy is strongly favored over an inverted hierarchy.

For several years now there has been keen interest in the use of A_4 as a finite flavor symmetry in the lepton sector, especially neutrino mixing. In particular, the empirically approximate tribimaximal mixing of the three neutrinos can be predicted. It is usually stated that either normal or inverted neutrino mass spectrum can be predicted.

We revisit these two questions in a minimal A_4 framework with only one A_4-**3** of Higgs doublets coupling to neutrinos and permitting only renormalizable couplings. For such a minimal model there is more predicitivity regarding neutrino masses.

Although the standard model was originally discovered using the criterion of renormalizability, it is sometimes espoused that renormalizability is not prerequisite in an effective lagrangian. Nevertheless, imposing renormalizability in the present case is more sensible because it does render the model far more predictive by avoiding the many additional parameters associated with higher-order irrelevant operators. Our choice of Higgs sector also minimizes the number of free parameters.

It is sufficiently important to emphasize the concept that every result mentioned in this talk would be impossible without imposing renormalizability.

There is a wikipedia profile of S. Weinberg stating that his 2nd best paper (the 1st is the standard model) is Physica, **96A,** 327 (1979) which suggested that renormalizability is unnecessary in a low-energy effective lagrangian.

Although it has been fruitful in low-energy QCD, heavy-quark effective theory and technicolor this idea is inappropriate to fundamental model building in particle phenomenology.

A_4 *symmetry*

The group A_4 is the order g=12 symmetry of a regular tetrahedron T and is a subgroup of the rotation group $SO(3)$. A_4 has irreducible representations which are three singlets $1_1, 1_2, 1_3$ and a triplet 3. In the embedding $A_4 \subset SO(3)$ the **3** of A_4 is identified with the adjoint **3** of $SO(3)$.

Since the only Higgs doublets coupling to neutrinos in our model are in a **3** of A_4, it is very useful to understand geometrically the three components of a **3**.

A regular tetrahedron has four vertices, four faces and six edges. Straight lines joining the midpoints of opposite edges pass through the centroid and form a set of three orthogonal axes. Regarding the regular tetrahedron as the result of cutting off the four odd corners from a cube, these axes are parallel to the sides of the cube. With respect to the regular tetrahedron, a vacuum expectation value (VEV) of the **3** such as $< \mathbf{3} >= v(1, 1, -2)$, as will be used,clearly breaks SO(3) to U(1) and correspondingly A_4 to Z_2, since it requires a rotation by π about the 3-axis to restore the tetrahedron.

At the same time, we can understand the appearance of tribimaximal mixing with matrix

$$U_{TBM} = \begin{pmatrix} -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix}, \qquad (1)$$

and our definitions are such that the ordering $\nu_{1,2,3}$ and $\nu_{\tau,\mu,e}$ satisfy

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{TBM} \begin{pmatrix} \nu_\tau \\ \nu_\mu \\ \nu_e \end{pmatrix} \qquad (2)$$

With respect to a side of the aforementioned cube, a face diagonal is at angle $\pi/4$ and the body diagonal is at angle $\tan^{-1}(\sqrt{\frac{1}{2}})$ with respect to the body diagonal. These two angles, together with $\theta_{13} = 0$ are the three corresponding to the matrix of Eq.(1). As we shall show this mixing occurs naturally for VEVs $< \mathbf{3} > \propto (1, 1, 1)$ and $< \mathbf{3} > \propto (1, -2, 1)$.

Assuming no CP violation, the Majorana matrix M_ν is real and symmetric and therefore of the form

$$M_\nu = \begin{pmatrix} A & B & C \\ B & D & F \\ C & F & E \end{pmatrix} \tag{3}$$

and is related to the diagonalized form by

$$M_{diag} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = U_{TBM} M_\nu U_{TBM}^T. \tag{4}$$

Substituting Eq.(1) into Eq.(4) shows that M_ν must be of the general form in terms of real parameters A, B, C:

$$M_\nu = \begin{pmatrix} A & B & C \\ B & A & C \\ C & C & A+B-C \end{pmatrix}, \tag{5}$$

which has eigenvalues

$$\begin{aligned} m_1 &= (A+B-2C) \\ m_2 &= (A+B+C) \\ m_3 &= (A-B). \end{aligned} \tag{6}$$

The observed mass spectrum corresponds approximately to $|m_1| = |m_2|$ which requires either $C = 0$ or $C = 2(A + B)$. For a normal hierarchy, $(A + B) = 0$ and $C = 0$. For an inverted hierarchy $A = B$ and $C = 0$ or $C = 4A$.

Now we study our minimal A_4 model to examine the occurrence of the Majorana matrix Eq.(5) and the eigenvalues Eq.(6).

Minimal A_4 model

We assign the leptons to (A_4, Z_2) irreps as follows

$$\left.\begin{array}{c}\begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L \\ \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \\ \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L\end{array}\right\} L_L(3,+1) \quad \begin{array}{ll} \tau_R^- \; (1_1,-1) & N_R^{(1)} \; (1_1,+1) \\ \mu_R^- \; (1_2,-1) & N_R^{(2)} \; (1_2,+1) \\ e_R^- \; (1_3,-1) & N_R^{(3)} \; (1_3,+1). \end{array} \tag{7}$$

The lepton lagrangian is

$$\mathcal{L}_Y^{(leptons)} = \frac{1}{2} M_1 N_R^{(1)} N_R^{(1)} + M_{23} N_R^{(2)} N_R^{(3)} \\ + \left[Y_1 \left(L_L N_R^{(1)} H_3 \right) + Y_2 \left(L_L N_R^{(2)} H_3 \right) \right. \\ + Y_3 \left(L_L N_R^{(3)} H_3 \right) \\ + Y_\tau \left(L_L \tau_R H_3' \right) + Y_\mu \left(L_L \mu_R H_3' \right) \\ \left. + Y_e \left(L_L e_R H_3' \right) \right] + \text{h.c.} \tag{8}$$

where $SU(2)$-doublet Higgs scalars are in $H_3(3, +1)$ and $H_3^{'}(3, -1)$.

The charged lepton masses originate from $< H_3^{'} >= (\frac{m_\tau}{Y_\tau}, \frac{m_\mu}{Y_\mu}, \frac{m_e}{Y_e})$ and are, to leading order, disconnected from the neutrino masses if we choose a flavor basis where the charged leptons are mass eigenstates. The N_R^i masses break the $L_\tau \times L_\mu \times L_e$ symmetry but change the charged lepton masses only by very small amounts $\propto Y^2 m_i / M_N$ at one-loop level.

The right-handed neutrinos have mass matrix

$$M_N = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_{23} \\ 0 & M_{23} & 0 \end{pmatrix}. \qquad (9)$$

We take the VEV of the scalar H_3 to be

$$< H_3 >= (V_1, V_2, V_3), \tag{10}$$

whereupon the Dirac matrix is

$$M_D = \begin{pmatrix} Y_1V_1 & Y_2V_3 & Y_3V_2 \\ Y_1V_3 & Y_2V_2 & Y_3V_1 \\ Y_1V_2 & Y_2V_1 & Y_3V_3 \end{pmatrix}. \tag{11}$$

The Majorana mass matrix M_ν is given by

$$M_\nu = M_D M_N^{-1} M_D^T. \tag{12}$$

We summarize some technical details we provided in arXiv:0806.1707:

Defining $x_1 \equiv Y_1^2/M_1$ and $x_{23} \equiv Y_2 Y_3/M_{23}$ we find a general symmetric form for M_ν and to ensure its texture gives tribimaximal mixing we find three equations corresponding to the mixing angles. We find no solution with any of $x_1, x_{23}, V_1, V_2, V_3$ vanishing. Non-trivial solution requires $V_1 = V_3$ and further requires $(2V_1 + V_2)(V_1 - V_2) = 0$.

The results of this lengthy calculation is that the only VEVs for H_3 which give TBM in the minimal renormalizable A_4 model are

$$< H_3 > \propto (1,1,1) \quad \text{or} \quad < H_3 > \propto (1,-2,1)$$

Consider first

$$< H_3 >= (V, V, V) \tag{13}$$

Careful comparison with the mass eigenstates reveals that $m_2 \gg m_1 = m_3 = 0$ so the wrong mass eigenstate (ν_2) is selected for the observed hierarchy, and so Eq.(13) is an unacceptable VEV for $< H_3 >$.

The only other VEV for the A_4-**3** is therefore

$$< H_3 >= (V, -2V, V), \tag{14}$$

which also gives tribimaximal mixing and the mass spectrum $m_3 \gg m_1 = m_2$ corresponding to a normal hierarchy. Eq.(14) provides the only acceptable VEV for $< H_3 >$.

An inverted hierarchy with $m_1 = m_2 \gg m_3$ is not possible within a minimal A_4 renormalizable model and so is disfavored. This means, for example, that neutrinoless double β-decay will require higher precision experiments.

Our conclusion is that contrary to previous discussions the A_4 model in a minimal form does favor the normal hierarchy. We have considered a more restrictive model based on A_4 than previously considered. The theory has been required to be renormalizable and the Higgs scalar content is the minimum possible.

We have required that the neutrino mixing matrix be of the tribimaximal form. We then find that the masses for the neutrinos are highly constrained and can be in a normal, not inverted hierarchy.

Most, if not all, previous A_4 models in the literature permit higher-order irrelevant non-renormalizable operators and their concomitant proliferation of parameters and hence allow a wide variety if possiblities for the neutrino masses. We believe the renormalizabilty condition is sensible for these flavor symmetries because of the higher predictivity.

The next step which is the subject of the rest of this talk is whether the present renornmalizable A_4 model can be extended to a renormalizable $T^{'}$ model. It is necessary but not sufficent condition for this that a successful renormalizable A_4 model, as presented here, exists.

4. T' symmetry.

The first use of the binary tetrahedral group T' in particle physics was by Case, Karplus and Yang in 1956 who were motivated to consider gauging a finite T' subgroup of $SU(2)$ in Yang-Mills theory. This led Fairbairn, Fulton and Klink (FFK) in 1964 to make an analysis of T' Clebsch-Gordan coefficients As a flavor symmetry, T' first appeared in 1994 motivated by the idea of representing the three quark families with the third treated differently from the first two. Since T' is the double cover of A_4, it was natural to suggest that T' be employed to accommodate quarks and simultaneously the established A_4 model building for tribimaximal neutrino mixing.

We shall discuss such a T' model with simplifications to emphasize the largest quark mixing, the Cabibbo angle, for which we shall derive an entirely new formula as an exact angle.

Recall that charged lepton masses arise from the vacuum expectation value

$$< H_3^{'} >= \left(\frac{m_\tau}{Y_\tau}, \frac{m_\mu}{Y_\mu}, \frac{m_e}{Y_e} \right) = (M_\tau, M_\mu, M_e) \tag{15}$$

where $M_i \equiv m_i/Y_i$ $(i = \tau, \mu, e)$. Neutrino masses and mixings come from the see-saw mechanism and the VEV

$$< H_3 >= V(1, -2, 1) \tag{16}$$

We shall now promote A_4 to $T^{'}$ keeping renormalizability and including quarks.

5. Minimal T' model

The left-handed quark doublets $(t,b)_L, (c,d)_L, (u,d)_L$ are assigned under $(T' \times Z_2)$ to

$$
\begin{array}{lll}
\begin{pmatrix} t \\ b \end{pmatrix}_L & \mathcal{Q}_L & (\mathbf{1_1}, +1) \\
\left.\begin{array}{l} \begin{pmatrix} c \\ s \end{pmatrix}_L \\ \begin{pmatrix} u \\ d \end{pmatrix}_L \end{array}\right\} & Q_L & (\mathbf{2_1}, +1)
\end{array}
\tag{17}
$$

and the six right-handed quarks as

$$
\begin{array}{lll}
t_R & & (\mathbf{1_1}, +1) \\
b_R & & (\mathbf{1_2}, +1) \\
\left.\begin{array}{l} c_R \\ u_R \end{array}\right\} & \mathcal{C}_R & (\mathbf{2_3}, -1) \\
\left.\begin{array}{l} s_R \\ d_R \end{array}\right\} & \mathcal{S}_R & (\mathbf{2_2}, +1)
\end{array}
\tag{18}
$$

We add only two new scalars $H_{1_1}(1_1, +1)$ and $H_{1_3}(1_3, +1)$ whose VEVs

$$< H_{1_1} >= m_t/Y_t \qquad < H_{1_3} >= m_b/Y_b \quad (19)$$

provide the (t, b) masses. In particular, no T' doublet $(2_1, 2_2, 2_3)$ scalars have been added. This allows a non-zero value only for Θ_{12}. The other angles vanish making the third family stable [#1].

[#1] As we shall discuss non-vanishing Θ_{23} and Θ_{13} are related to (d, s) masses.

The allowed quark Yukawa and mass terms are

$$
\begin{aligned}
\mathcal{L}_Y^{(quarks)} = & Y_t(\{\mathcal{Q}_L\}_{\mathbf{1_1}}\{t_R\}_{\mathbf{1_1}}H_{\mathbf{1_1}}) \\
& + Y_b(\{\mathcal{Q}_L\}_{\mathbf{1_1}}\{b_R\}_{\mathbf{1_2}}H_{\mathbf{1_3}}) \\
& + Y_{\mathcal{C}}(\{Q_L\}_{\mathbf{2_1}}\{\mathcal{C}_R\}_{\mathbf{2_3}}H_{\mathbf{3}}^{'}) \\
& + Y_{\mathcal{S}}(\{Q_L\}_{\mathbf{2_1}}\{\mathcal{S}_R\}_{\mathbf{2_2}}H_{\mathbf{3}}) \\
& + \text{h.c.}
\end{aligned}
\qquad (20)
$$

The use of $T^{'}$ singlets and doublets #2 for quark families in Eqs.(17,18) permits the third family to differ from the first two and thus make plausible the mass hierarchies $m_t \gg m_b$, $m_b > m_{c,u}$ and $m_b > m_{s,d}$.

#2 It is discrete anomaly free. We thank the UF-Gainesville group for discussions.

6 The Cabibbo angle

The nontrivial (2×2) quark mass matrices (c,u) and (s,d) will be respectively denoted by $U^{'}$ and $D^{'}$ and calculated using the $T^{'}$ Clebsch-Gordan coefficients of Fairbairn, Fulton and Klink. Dividing out $Y_{\mathcal{C}}$ and $Y_{\mathcal{S}}$ in Eq.(20) gives U and D matrices $(\omega = e^{i\pi/3})$

$$U \equiv \left(\frac{1}{Y_{\mathcal{C}}}\right) U^{'} = \begin{pmatrix} \sqrt{\frac{2}{3}}\omega^2 M_\tau & \frac{1}{\sqrt{3}} M_e \\ -\frac{1}{\sqrt{3}}\omega^2 M_e & \sqrt{\frac{2}{3}} M_\mu \end{pmatrix} \tag{21}$$

$$D \equiv \left(\frac{1}{Y_{\mathcal{S}}}\right) D^{'} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -2\sqrt{\frac{2}{3}}\omega \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}\omega \end{pmatrix} \tag{22}$$

Let us first consider U of Eq.(21). Noting that $m_\tau > m_\mu \gg m_e$ we may simplify U by setting the electron mass to zero, $M_e = 0$. This renders U diagonal leaving free the c, u, τ and μ masses. This leaves only the matrix D in Eq.(22) which predicts both Θ_{12} and (m_d^2/m_s^2). The hermitian square $\mathcal{D} \equiv DD^\dagger$ is

$$\mathcal{D} \equiv DD^\dagger = \left(\frac{1}{3}\right) \begin{pmatrix} 9 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{pmatrix} \tag{23}$$

which leads by diagonalization to a formula for the Cabibbo angle

$$\tan 2\Theta_{12} = \left(\frac{\sqrt{2}}{3}\right) \tag{24}$$

or equivalently $\sin\Theta_{12} = 0.218..$ close to the experimental value [#3] $\sin\Theta_{12} \simeq 0.227$.

Our result of an exact angle for Θ_{12} can be regarded as on a footing with the tribimaximal values for neutrino angles θ_{ij}.

[#3] Experimental results are from PDG2006; see references therein.

Note that the tribimaximal θ_{12} presently agrees with experiment within one standard deviation (1σ). On the other hand, our analagous exact angle for Θ_{12} differs from experiment already by 9σ which is probably a reflection of the fact that the experimental accuracy for Θ_{12} is $\sim 0.5\%$ while that for θ_{12} is $\sim 6\%$.

It is thus very important to acquire better experimental data on θ_{12}, θ_{23} and θ_{13} to detect their similar deviation from the exact angles predicted by TBM. Our result for (m_d^2/m_s^2) from Eq.(23) is exactly 0.288.. compared to the central experimental value $\simeq 0.003$ in a simplified model whose generalization to an extended scalar sector including $T^{'}$ doublets can avoid $\Theta_{23} = \Theta_{13} = 0$ and thereby change (m_d^2/m_s^2) due to mixing of (d, s) with the b quark.

This $T' \times Z_2$ extension of the standard model is an first step to tying the quark and lepton sectors together, providing calculability, and at the same time reducing the number of standard model parameters. The ultimate goal would be to understand the origin of this discrete symmetry. Since gauge symmetries can break to discrete symmetries, and gauge symmetries arise naturally from strings, perhaps there is a clever construction of our model with its fundamental origin in string theory.

7. Higgs boson decay

The use of $(T^{'} \times Z_2)$ led above to a successful prediction for the Cabibbo angle:

$$\tan 2\Theta_{12} = \left(\frac{\sqrt{2}}{3} \right) \tag{25}$$

The same model leads to an even more striking prediction when we study the messenger scalar linking the charged leptons $(\tau, \mu, e)_L$ to the U-type quarks contained in Q_L and $\mathcal{C}_R(2_3, -1)$ $[(c, u)_R]$. This gives rise to an expression for the ratio of branching ratios for Higgs decay

$$r = \left(\frac{\Gamma(H \longrightarrow \tau^+\tau^-)}{\Gamma(H \rightarrow \mu^+\mu^-)} \right) \tag{26}$$

We recall that in the minimal standard model the two body decays in Eq.(26) satisfy at tree level

$$(r)_{SM} = \left(\frac{m_\tau^2 (1 - 4(m_\tau^2/M_W^2))^{3/2}}{m_\mu^2 (1 - 4(m_\mu^2/M_W^2))^{3/2}} \right) \simeq 280 \tag{27}$$

In the $(T^{'} \times Z_2)$ model the messenger scalar $H_3^{'}(3, -1)$ which couples both to neutrinos and to U-type quarks in the first two generations provides a large change from Eq.(27) to give

$$(r)_{T'} = \left(\frac{m_\tau^2 (1 - 4(m_\tau^2/M_W^2))^{3/2}}{m_\mu^2 (1 - 4(m_\mu^2/M_W^2))^{3/2}} \right) \left(\frac{m_u^2}{m_c^2} \right) \simeq 0.001 \tag{28}$$

The change in r from the minimal standard model to $T^{'}$ flavor symmetry is more than a factor 250,000!!!

Since the Higgs boson and its decay branching ratios are targets of opportunity for the L.H.C., Eq.(28) can provide a smoking gun for such a quark-lepton relationship arising from $(T^{'} \times Z_2)$ flavor symmetry.

We may ask about the predictions for the separate numerator and denominator in r. Further study reveals that only the ration r is firmly predicted because the separate decay modes depend on the precise identification of the light Higgs doublet among the scalars of the $(T^{'} \times Z_2)$ model. In the ratio r, however, this uncertainty cancels out.

It will be amusing to see from experimental data about Higgs boson decays whether Nature chooses such a broken symmetry in relating quarks and leptons.

If such an effect is observed, and it would seem difficult to miss once the Higgs boson is discovered, it will provide strong evidence, a smoking gun, for this alternative to grand unification.

8. Neutrino Mixing Angles

The three neutrino mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ are empirically consistent with the TBM values. However, as the experimental accuracy improves, this situation may change. Thus, it is of considerable interest to predict quantitatively what departures from the TBM values

$$\theta_{12} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right), \quad \theta_{23} = (\pi/4), \quad \theta_{13} = 0 \tag{29}$$

are to be expected? We are delighted to report that the $(T^{'} \times Z_2)$ model allows one to address this question by relating the perturbations around TBM

$$\theta_{ij} = (\theta_{ij})_{TBM} + \epsilon_k, \tag{30}$$

To analyze the relationship between the perturbations in Eq.(30) and the Cabibbo angle will require, as we shall see, very interesting $T^{'}$ algebra sometimes arriving at astonishingly simple expressions.

First we recall a few salient points about the model based on A_4 symmetry. The only important scalar for the present analysis is the triplet $H_3(3, +1)$ whose vacuum expectaion value was taken as

$$< H_3 >= (V_1, V_2, V_3) = V(1, -2, 1) \quad (31)$$

which led to the TBM neutrino mixing in Eq.(1). We consider the perturbation

$$< H_3 >= (V_1^{'}, V_2^{'}, V_3^{'}) = V^{'}(1, -2 + b, 1 + a) \quad (32)$$

The down-quark (2×2) mass matrix for the first two families (s, d) is perturbed to

$$D \equiv \left(\frac{1}{V'Y_{\mathcal{S}}}\right) D' = \begin{pmatrix} \frac{1}{\sqrt{3}} & (-2+b)\sqrt{\frac{2}{3}}\omega \\ \sqrt{\frac{2}{3}}(1+a) & \frac{1}{\sqrt{3}}\omega \end{pmatrix} \tag{33}$$

The hermitian square $\mathcal{D} \equiv DD^\dagger$ is

$$\mathcal{D} = \left(\frac{1}{3}\right) \begin{pmatrix} 9-8b & \sqrt{2}(-1+a+b) \\ \sqrt{2}(-1+a+b) & 3+4a \end{pmatrix}, \tag{34}$$

whose eigenvalues are

$$\lambda_\pm = (6 \pm \sqrt{11}) + 2a\left(1 \mp \frac{4}{\sqrt{11}}\right) - 2b\left(2 \pm \frac{7}{\sqrt{11}}\right) \tag{35}$$

An eigenvector (α, β) satisfies

$$\left(\frac{\beta}{\alpha}\right) = \left(\frac{3-\sqrt{11}}{\sqrt{2}}\right)\left[1 - \frac{a}{\sqrt{11}} + \frac{b}{\sqrt{11}}\right] \tag{36}$$

whose normalization $N^{-2} = 1 + \beta^2/\alpha^2$ gives

$$\sin\Theta_{12} = \sqrt{\left(\frac{1}{2} - \frac{3}{2\sqrt{11}}\right)}\left(1 - \frac{3+\sqrt{11}}{22}(a-b)\right) \tag{37}$$

From this one finds

$$\cos 2\Theta_{12} = \left(\frac{3}{\sqrt{11}}\right)\left(1 + \frac{2}{33}(a-b)\right) \quad (38)$$

and

$$\sin 2\Theta_{12} = \left(\frac{2}{\sqrt{11}}\right)\left(1 - \frac{3}{11}(a-b)\right), \quad (39)$$

whence

$$\tan 2\Theta_{12} = (\sqrt{2})/3\left(1 - \frac{1}{3}(a-b)\right) \quad (40)$$

which is a suprisingly simple generalization of the $a = b = 0$ case!

Making the perturbations defined above, one has, at first order, $s_{12} = \sqrt{\frac{1}{3}}(1 + \sqrt{2}\epsilon_3)$; $c_{12} = \sqrt{\frac{2}{3}}(1 - \epsilon_3/\sqrt{2})$; $s_{23} = \sqrt{\frac{1}{2}}(1 + \epsilon_1)$; $c_{23} = \sqrt{\frac{1}{2}}(1 - \epsilon_1)$. ;$s_{13} = \epsilon_2$; $c_{13} = 1$;

We obtain six equations from the (3×3) symmetric matrix. In the δm_1 of (I) - (III) a common (unpredicted) normalization factor has been omitted.

- (I) $\delta m_1 = (2 + y)(a - 2b)$
- (II) $\delta m_2 = 0$
- (III) $\delta m_3 = -3y(a - 2b)$
- (IV) $\epsilon_2 = -\sqrt{2}\epsilon_1$
- (V) $a = 6\epsilon_1 = -3\sqrt{2}\epsilon_2$
- (VI) $(a + b) = \left(\frac{3}{\sqrt{2}}\frac{2+y}{1-y}\right)\epsilon_3$

The result (IV) provides a prediction from $T^{'}$ that

$$\theta_{13} = \sqrt{2}\left(\frac{\pi}{4} - \theta_{23}\right) \qquad (41)$$

which interestingly links any non-zero value for θ_{13} to the departure of the atmospheric neutrino mixing angle θ_{23} from maximal mixing with $\theta_{23} = \pi/4$. This is our most definite prediction from $T^{'}$, independent of phenomenoligical input.

With regard to quark and lepton masses, the flavor symmetry leaves them (so far?) as enigmatic as before, basically as free parameters just as for the standard model.

SUMMARY

1. Renormalizability and simplification of $(A_4 \times Z_2)$ then $(T^{'} \times Z_2)$ models lead to:

Cabibbo angle formula

$$\tan 2\Theta_{12} = \left(\frac{\sqrt{2}}{3} \right)$$

In Higgs boson decay

$$r = \left(\frac{\Gamma(H \longrightarrow \tau^+ \tau^-)}{\Gamma(H \rightarrow \mu^+ \mu^-)} \right) \tag{42}$$

$$(r)_{T'} = (r)_{SM} \left(\frac{m_u^2}{m_c^2} \right) \simeq 0.001 \tag{43}$$

which represents a reduction by a 250,000 relative to the standard model.

Neutrino mixing angles are predicted to satisfy

$$\theta_{13} = \sqrt{2}\left(\frac{\pi}{4} - \theta_{23}\right) \tag{44}$$

General conclusion, further study of the binary tetrahedral group is merited, especially if the above predictions will be supported by future experiments.

Thank you for your attention