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INTEGRABILITY IN LARGE N SUPERCONFORMAL YANG MILLS THEORY

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SPIN CHAIN METHODS

- INTEGRABLE HAMILTONIANS AND THEIR EIGENVALUES AND EIGENSTATES
- YANGIAN SYMMETRY ALGEBRA
- R-MATRIX : BETHE ANSATZ

SOLVABILITY IN

- $N=4$ $D=4$ $SU(N)$ GAUGE THEORY
- TYPE IIB SUPERSTRING ON $AdS_5 \times S^5$

$$\begin{tikzcd}
\text{d-DIMENSIONAL CLASSICAL STAT. MECH.} \arrow[rr] \arrow[dr] & & \text{SPACE (d-1)-DIMENSIONAL CONTINUUM QFT} \\
& \text{(d-1)-DIMENSIONAL LATTICE QFT} \arrow[ur] &
\end{tikzcd}$$

E.G. $d=2$

$$\begin{tikzcd}
\text{BAXTER 8-VERTEX MODEL} \arrow[rr] \arrow[dr] & & \text{SINE-GORDON QFT} \\
& \text{XYZ SPINCHAIN} \arrow[ur] &
\end{tikzcd}$$

- $$Z_{8\text{-VERTEX}} = \sum_{N_j} e^{-\sum_{j=1}^{8} \epsilon_j N_j} = \sum_{\vec{\alpha}_1 \ldots \vec{\alpha}_L} \langle \vec{\alpha}_1 | \hat{T} | \vec{\alpha}_2 \rangle \ldots \langle \vec{\alpha}_L | \hat{T} | \vec{\alpha}_1 \rangle$$
$$= \mathrm{Tr}\, \hat{T}^{L}$$

- TRANSFER MATRIX $\hat{T}(v) = \mathrm{tr} \prod_{i=1}^{L} \sum_{j=1}^{4} \omega_j \hat{\sigma}^j(i) \sigma^j$

- $$\frac{\partial}{\partial v} \ln \hat{T} \Big|_{v=\zeta} \sim H_{XYZ} \sim J^A \sum_{i=1}^{L} \sigma_i^A \sigma_{i+1}^A$$

- $$I_{\text{SINE GORDON}} = \int dt \int dx \left(\frac{1}{16\pi} (\partial_\nu \varphi)^2 - 2\mu \cos\beta\varphi \right)$$
(SCALING LIMIT)

HEISENBERG $XXX_{\frac{1}{2}}$ MODEL

$$H_{XXX_{\frac{1}{2}}} = K \sum_{i=1}^{L-1} \left(J_i^A J_{i+1}^A + \frac{1}{4} \right)$$

$$[J_i^A, J_j^B] = \varepsilon_{ABC} J_i^C \delta_{ij},$$

$$J_j^A \equiv \frac{\sigma^A}{2i} \quad SU(2)$$

$$\boxed{K = \frac{g_{YM}^2 N}{4\pi^2}}$$

THE GOAL OF INTEGRABLE METHODS IS TO DIAGONALIZE THE HAMILTONIAN.

THE BASIS VECTORS SPAN 2^L-DIMENSIONAL HILBERT SPACE.

WE COULD TRY TO DO THIS DIRECTLY BY HAND (OR COMPUTER)

E.G. $L=2$ SITES

BASIS STATES

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$L = 2$

$$H = 2K \left(J_1^A J_2^A + \tfrac{1}{4} \right)$$

$$= K\left(-\tfrac{1}{2}\sigma_1^A \sigma_2^A + \tfrac{1}{2}\right) = -K \begin{pmatrix} 0&0&0&0 \\ 0&-1&1&0 \\ 0&1&-1&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= -K \left(P_{12} - I_1 \otimes I_2 \right)$$

PERMUTATION OPERATOR

$$P_{12} = \tfrac{1}{2}\left(I_1 \otimes I_2 + \sigma_1^A \otimes \sigma_2^A \right)$$

$$P_{12}\,(a \otimes b) = b \otimes a$$ PERMUTES THE TENSOR PRODUCT

E.G. $P_{12} \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \cancel{\tfrac{1}{2} \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}}$

$$+ \tfrac{1}{2}\left(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3\right) \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-i\end{pmatrix} \otimes \begin{pmatrix}i\\0\end{pmatrix} + \cancel{\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\-1\end{pmatrix}}$$

$$= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}.$$

EIGENSTATES	EIGENVALUES
$\vert+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$	0
$\vert 0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$	0
$\vert-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$	0
$\vert S\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$	$2K$

$$\boxed{\text{SYM}_4:\ \ |+\rangle \sim \mathrm{Tr}\, ZZ,\quad |0\rangle \sim \mathrm{Tr}\, ZW,\quad |-\rangle \sim \mathrm{Tr}\, WW}$$

BUT FOR A LARGER NUMBER OF SITES L, FINDING THE EIGENVALUES BY HAND BECOMES DIFFICULT.

LET'S LOOK AT THE SYMMETRIES:

- TOTAL SPIN VARIABLES $J^A = \sum_{i=1}^{L} J_i^A$ ARE ORDINARY SU(2) GENERATORS
  $[J^A, J^B] = \varepsilon_{ABC} J^C$
- $[H_{XXX_{\frac{1}{2}}}, J^A] = 0$
  E.G. $[J_1^C J_2^C, J_1^A + J_2^A] = \varepsilon_{CAB}(J_1^B J_2^C + J_1^C J_2^B) = 0$
  INDEPENDENT OF THE REPRESENTATION OF J_i^A.
- WE CAN WRITE THE HAMILTONIAN IN TERMS OF THE TWO-SITE QUADRATIC CASIMIR $(J_1^A + J_2^A)(J_1^A + J_2^A) = J^A J^A$:

E.G. $H = K(J_1^A J_2^A + 1/4) = \frac{K}{2}(J^A J^A + 2)$

$$= \frac{K}{2}(-J(J+1) + 2).$$

$(J_1^A)^2 + (J_2^A)^2 + 2 J_1^A J_2^A$

$-\frac{1}{4}(\sigma^A)^2 = -3/4$

$J = 0, \frac{1}{2}, 1, 3/2, \ldots$

$|S\rangle \quad |+\rangle, |0\rangle, |-\rangle$

LET $J^A = J^3, J^{\pm}$

$J^{\pm}$ ARE RAISING AND LOWERING GENERATORS WITHIN A MULTIPLET:

$$\begin{array}{ccc} \downarrow \uparrow & |+\rangle & \\ & |0\rangle & |S\rangle \\ J^- \; J^+ & |-\rangle & \end{array}$$

WHAT ABOUT GOING BETWEEN MULTIPLETS?

ARE THERE ANY MORE SYMMETRY GENERATORS?

YES.

SINCE WE HAVE CONSIDERED AN OPEN TOPOLOGY FOR THE SPIN CHAIN, INSTEAD OF PERIODIC BOUNDARY CONDITIONS

WE CAN CONSTRUCT THE <u>YANGIAN</u> GENERATORS $\mathcal{J}_n^A$, $n = 0, 1, \ldots, L-1$

$\mathcal{J}_n^A$ ACTS ON $n+1$ SITES AT A TIME.

$\mathcal{J}_0^A = J^A = \sum_i J_i^A$ LOCAL

$\mathcal{J}_1^A = Q^A = \varepsilon_{ABC} \sum_{i<j} J_i^B J_j^C$ NON-LOCAL

ETC.

WHAT IS THE ALGEBRA?

WHAT ARE THE COMMUTATION RELATIONS?

- HOW DO THE GENERATORS ACT ON THE STATES?

ARE THE NEW GENERATORS SYMMETRIES?

FOR THE $L=2$ EXAMPLE,

$$Q^3 = \tfrac{1}{2}\left(J_1^- J_2^+ - J_1^+ J_2^- \right)$$
$$Q^+ = J_1^+ J_2^3 - J_1^3 J_2^+$$
$$Q^- = J_1^3 J_2^- - J_1^- J_2^3$$

THE BILOCAL GENERATORS Q^A ACT ON THE $L=2$ EIGENSTATES AS

$$Q^{3,+}|+\rangle = 0$$
$$Q^-|+\rangle = \tfrac{1}{2}|S\rangle$$

$$Q^3|0\rangle = -\tfrac{1}{2}|S\rangle$$
$$Q^\pm|0\rangle = 0$$

$$Q^{3,-}|-\rangle = 0$$
$$Q^+|-\rangle = -\tfrac{1}{2}|S\rangle$$

$$Q^\pm|S\rangle = 0$$
$$Q^3|S\rangle = \tfrac{1}{2}|0\rangle$$

$$\begin{tikzcd}
|+\rangle \arrow[dr, "Q^-"] & \\
|0\rangle \arrow[r, "Q^3"] & |S\rangle \\
|-\rangle \arrow[ur, "Q^+"'] &
\end{tikzcd}$$

COMMUTATION WITH THE HAMILTONIAN IS

$$[H_{xxx\frac{1}{2}}, Q^A] = \frac{K}{2}(J_1^A - J_L^A)$$

SINCE

$$[H_{12}, Q^A_{12}] = K\,\epsilon_{ABC}\,[J_1^D J_2^D, J_1^B J_2^C]$$
$$= K\,(J_1^C J_2^C J_2^A - J_1^A J_2^C J_2^C - J_1^A J_1^B J_2^B + J_1^B J_1^B J_2^A)$$
$$= \frac{K}{2}(J_1^A - J_2^A)$$

WHERE THIS USES THE REPRESENTATION

AND

$$[H_{xxx\frac{1}{2}}, Q^A] = \sum_{i=1}^{L-1} [H_{i,i+1}, Q^A_{i,i+1}] = \frac{K}{2}\sum_{i=1}^{L-1} q^A_{i,i+1} = \frac{K}{2} q^A$$

$q^A_{i,i+1} = J_i^A - J_{i+1}^A$ IS THE DIFFERENCE OPERATOR,

$q^A = J_1^A - J_L^A$. A LATTICE VERSION OF A TOTAL DERIVATIVE.

THE CROSS TERMS VANISH DUE TO $[H_{i,i+1}, J_i^A + J_{i+1}^A] = 0$.

FOR FINITE CHAINS, THE YANGIAN COMMUTES WITH THE HAMILTONIAN UP TO EDGE EFFECTS.

FOR CHAINS OF INFINITE LENGTH WHERE WE IGNORE A TOTAL DERIVATIVE, THE YANGIAN IS AN EXACT SYMMETRY.

$$[H_{xxx\frac{1}{2}}, \mathcal{C}(\mathcal{J}^A_n)] = 0$$

CASIMIR OF YANGIAN IS DEFINED FOR PERIODIC BOUNDARY CONDITIONS

YANGIAN ALGEBRA $Y(G)$

- $[J^A, J^B] = f^{AB}{}_C J^C$ $\qquad J^A \in G$ SEMI-SIMPLE LIE GROUP
- $[J^A, Q^B] = f^{AB}{}_C Q^C$

HIGHER YANGIAN GENERATORS ARISE IN COMMUTATORS OF Q^A's:

$$[Q^A, Q^B] = f^{AB}{}_C \mathcal{J}_2^C + \cdots$$

- SERRE RELATIONS $\quad [Q^A, [Q^B, J^C]] + \cdots$

AS $L \to \infty$, $Y(G)$ IS AN INFINITE-DIMENSIONAL ALGEBRA. IT GENERATES THE SAME EQUIVALENCE RELATION AS THE <u>KAC-MOODY LOOP ALGEBRA (KMA)</u>:

$$[M^A_n, M^B_m] = f^{AB}{}_C M^C_{n+m} \qquad n = 0, 1, 2, \ldots$$

THE INFINITESIMAL TRANSFORMATIONS GENERATED BY $Y(G)$ AND THE <u>KMA</u> ARE LINEAR COMBINATIONS OF EACH OTHER (BUT WITH FIELD DEPENDENT PARAMETERS).

BOTH ALGEBRAS ARE INFINITE-DIMENSIONAL AND NON-ABELIAN.

THE YANGIAN IS A HOPF ALGEBRA.
IT HAS A NATURAL RECIPE FOR DEFINING TENSOR PRODUCTS OF REPRESENTATIONS (MORE LATER)

IT WAS INTRODUCED BY V. DRINFELD IN 1985.
KMA IDENTIFIED AS SIGMA-MODEL (PCM) SYMMETRY IN 1981 L.D.
(SIMILAR TO TYPE IIB STRING ON $AdS_5 \times S^5$)
BENA, POLCHINSKI ROIBAN 2003

HOW DOES THE SYMMETRY ALGEBRA HELP US FIND THE EIGENVALUES OF THE SPIN CHAIN HAMILTONIAN FOR ANY L?

WHY IS THE SPIN CHAIN INTEGRABLE?

- AN INTEGRABLE HAMILTONIAN MEANS THAT IT IS ONE OF A SET OF L-1 COMMUTING HAMILTONIANS

$$[H^{(n)}, H^{(m)}] = 0$$

EG.

$$H^{(1)} = H_{XXX_{\frac{1}{2}}} = K \sum_i (J_i^A J_{i+1}^A + 1/4)$$

$$H^{(2)} = \varepsilon_{ABC} \sum_i J_i^A J_{i+1}^B J_{i+2}^C$$

ETC.

(LOCAL)

THESE ABELIAN HAMILTONIANS ARE CASIMIRS OF THE YANGIAN

$$[H^{(n)}, J^A] = 0$$
$$[H^{(n)}, Q^A] \sim 0$$

THE PROCEDURE FOR DIAGONALIZING THESE HAMILTONIANS IS CALLED THE ALGEBRAIC BETHE ANSATZ (ABA) A.K.A. THE INVERSE SCATTERING METHOD.

AN IMPORTANT INGREDIENT OF THIS METHOD IS AN R-MATRIX WHICH IS A SOLUTION TO THE YANG-BAXTER RELATION (YBR)

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}$$

THE HAMILTONIAN EG. $H = \sum_i H_{i,i+1}$ CAN BE DERIVED IN TERMS OF THE R-MATRIX. THE ENERGY EIGENVALUES CAN BE FOUND DUE TO <u>YBR</u>.

THE YANG-BAXTER RELATION FOR A UNIVERSAL R-MATRIX ENCODES THE YANGIAN ALGEBRA.

LET'S SEE HOW THE COPRODUCT OF THE YANGIAN HOPF ALGEBRA ALLOWS US TO CONSTRUCT A SPIN CHAIN REPRESENTATION FROM A SINGLE SPIN REPRESENTATION:

- ONE SERRE RELATION IS

$$[Q^A,[Q^B,J^C]]+[Q^B,[Q^C,J^A]]+[Q^C,[Q^A,J^B]]$$
$$=\frac{1}{24}f^{ADK}f^{BEL}f^{CFM}f_{KLM}\underbrace{\{J_D,J_E,J_F\}}_{\text{SYMMETRIZED PRODUCT}}$$

OTHER DEFINING RELATIONS:

$$[J^A,J^B]=f^{AB}{}_C J^C$$
$$[J^A,Q^B]=f^{AB}{}_C Q^C$$

SINGLE SPIN REPRESENTATION: (L=1)

$$J^A=J^A_1$$
$$Q^A=0$$

CHAIN OF SPINS REPRESENTATION:

$$J^A=\sum_i J^A_i$$
$$Q^A=\sum_{i<j} f^A{}_{BC}J^B_i J^C_j$$

MULTI SPIN REPRESENTATION:

FOR AN ALGEBRA A,

A COPRODUCT IS A MAP $\Delta: A \to A \otimes A$

THAT IS A HOMOMORPHISM OF ALGEBRAS.

IT MAPS SINGLE SPIN REPRESENTATIONS

TO TWO-SPIN REPRESENTATIONS, ETC.

FOR $A = Y(G)$, THE EXPLICIT FORMULA

FOR THE COPRODUCT IS

$$\Delta J^A = J^A \otimes 1 + 1 \otimes J^A$$

$$\Delta Q^A = Q^A \otimes 1 + 1 \otimes Q^A + f^A{}_{BC}\, J^B \otimes J^C$$

GIVEN A SINGLE SPIN REPRESENTATION $J^A = J_1^A$, $Q^A = 0$,

THE TWO-SPIN REPRESENTATION is

$$J^A = \Delta J^A = J_1^A \otimes 1 + 1 \otimes J_2^A = J_1^A + J_2^A$$

$$Q^A = \Delta Q^A = f^A{}_{BC}\, J_1^B \otimes J_2^C = f^A{}_{BC}\, J_1^B J_2^C$$

FROM A ONE-SITE AND TWO-SITE REPRESENTATION, THE THREE-SPIN REPRESENTATION IS

$$J^A = (J_1^A + J_2^A) \otimes 1 + 1 \otimes J_3^A = J_1^A + J_2^A + J_3^A$$

$$\begin{aligned} Q^A &= f^A{}_{BC}\, J_1^B \otimes J_2^B \otimes 1 \\ &\quad + 0 + f^A{}_{BC} (J_1 + J_2)^B \otimes J_3^C \\ &= f^A{}_{BC} \left(J_1^B J_2^C + J_2^B J_3^C + J_1^B J_3^C \right) \\ &= f^A{}_{BC} \sum_{i<j} J_i^B J_j^C . \end{aligned}$$

FOR THE ALGEBRA $A = Y(G)$

THE UNIVERSAL R-MATRIX R IS AN ELEMENT IN $A \otimes A$

THE YANG BAXTER RELATION HOLDS IN $A \otimes A \otimes A$

WHERE $R_{12} = R \otimes I$, $R_{23} = I \otimes R$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

FADDEEV REVIEWS
hep-th/9605187
hep-th/9404013

THE ALGEBRA $Y(G)$ HAS REPRESENTATIONS $\rho(a, u)$

$\uparrow$ DISCRETE $\uparrow$ CONTINUOUS PARAMETER

Cf: $\tilde{\rho}(a, u) = T^A \otimes u^n \sim T^A_n$ REPRESENTATION OF LOOP ALGEBRA $\widehat{SU(2)}$

$$[T^A_n, T^B_m] = \varepsilon_{ABC} T^C_{n+m}$$

WHERE $T^A_0 = T^A$ is $(2a+1)$-DIMENSIONAL SU(2) REPRESENTATION

SINCE R IS AN ELEMENT OF $Y(G) \otimes Y(G)$, ITS REPRESENTATION HAS THE FORM

$$R(u-v) = (\rho(a,u) \otimes \rho(b,v))\, R$$

FOR $G = SU(2)$, $a, b = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

EG. $$R(u-v) = (\rho(\tfrac{1}{2}, u) \otimes \rho(\tfrac{1}{2}, v))\, R$$
$$= (u-v)\, \underset{\uparrow}{I_n} \otimes \underset{\uparrow}{I_m} + \frac{i}{2} (I_n \otimes I_m + \sigma^A_n \otimes \sigma^A_m)$$

DEFINED ON TWO SITES

A SOLUTION OF THE YBR

(RELEVANT FOR $XXX_{\frac{1}{2}}$ MODEL) IS

$$\mathcal{R}(u) = (u + i/_2)\, I_n \otimes I_m + \frac{i}{2}\, \sigma^A_n \otimes \sigma^A_m = u\, I_n \otimes I_m + i P_{nm}$$

↑ UNIVERSAL R-MATRIX REPRESENTATION

$$= \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}$$

$$\underline{R_{12}(u)} = \mathcal{R}(u) \otimes I = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a &&&&&&& \\ & a &&&&&& \\ && b && c &&& \\ &&& b && c && \\ && c && b &&& \\ &&& c && b && \\ &&&&&& a & \\ &&&&&&& a \end{pmatrix}$$

12 LABELS THE SITES

↑ $\boxed{8 \times 8}$ R-matrix $R_{12}(u)$

$a = u + i$
$b = u$
$c = i$

$$\underline{R_{23}(u)} = I \otimes \mathcal{R}(u) = \begin{pmatrix} a &&&&&&& \\ & b & c &&&&& \\ & c & b &&&&& \\ &&& a &&&& \\ &&&& a &&& \\ &&&&& b & c & \\ &&&&& c & b & \\ &&&&&&& a \end{pmatrix}$$

$$\underline{R_{13}(u)} = P_{23}\, R_{12}(u)\, P_{23}$$

$$P_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \frac{1}{2}\left(I \times I + \sigma^A \otimes \sigma^A\right)$$

YBR
$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)$$

A MATRIX EQUATION

THE HAMILTONIAN AND ITS EIGENVALUES ARE FOUND FROM

- THE MONODROMY MATRIX (SCATTERING DATA)

$$T_a(u) = L_{La}(u)\ldots L_{1a}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$
$$= u^L - u^{L-1} J^A \sigma_a^A - 2i u^{L-2} Q^A \sigma_a^A + \cdots$$

WHERE THE LAX OPERATOR IS $L_{na}(u) \equiv R_{na}(u - i/2)$.

$\uparrow$ AUXILIARY SPACE

- THE TRACE OF THE MONODROMY MATRIX (TRANSFER MATRIX).

$$t(u) = \mathrm{Tr}\; T_a(u) = A(u) + D(u).$$

- $$H_{XXX_{\frac{1}{2}}} = -\frac{\kappa}{2}\left(i \frac{d}{du} \ln t(u)\Big|_{u=\frac{i}{2}} - L \right) = -\frac{\kappa}{4}\sum_i \left(\sigma_i^A \sigma_{i+1}^A - 1\right)$$

- YBR FOR MONODROMY

$$R_{a_1 a_2}(u-v)\, T_{a_1}(u) T_{a_2}(v) = T_{a_2}(v) T_{a_1}(u) R_{a_1 a_2}(u-v)$$
$$t(u)\, t(v) = t(v)\, t(u)$$

WHICH IMPLIES RELATIONS AMONG $A(u), B(u), C(u), D(u)$.

THESE CAN BE USED TO DIAGONALIZE $t(u)$.

LET $|Z^L\rangle = \binom{1}{0} \otimes \ldots \otimes \binom{1}{0}$.

THE BETHE STATE

$$|u_1, \ldots u_M\rangle = B(u_1) \ldots B(u_M) |Z^L\rangle = |Z^{L-M} W^M\rangle$$

IS AN EIGENSTATE OF THE TRANSFER MATRIX

$$t(u) |u_1, \ldots u_M\rangle = \Lambda(u; u_1, \ldots u_M) |u_1 \ldots u_M\rangle$$

WHERE THE EIGENVALUE IS

$$\Lambda(u; u_1, \ldots u_M) = \left(u + \tfrac{i}{2}\right)^L \prod_{k=1}^{M} \frac{u - u_k - i}{u - u_k} + \left(u - \tfrac{i}{2}\right)^L \prod_{k=1}^{M} \frac{u - u_k + i}{u - u_k}$$

IF THE RAPIDITIES $u_1, \ldots, u_M$ ARE DISTINCT AND SATISFY THE BETHE ANSATZ EQUATIONS

$$\left(\frac{u_j + \frac{i}{2}}{u_j - i/2} \right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} .$$

THE HAMILTONIAN EIGENVALUE IS

$$\gamma = \frac{t}{2} \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}} .$$

THE MOMENTUM EIGENVALUE IS

$$P = -i \sum_{j=1}^{m} \ln \frac{u_j + \frac{i}{2}}{u_j - i/2}$$

WHERE THE MOMENTUM OPERATOR IS

$$-i \ln i^{-L} t\left(\tfrac{i}{2}\right).$$

THE ABELIAN HAMILTONIANS ARE

$$H^{(n)} \sim \frac{\partial^n}{\partial u^n} \ln t(u) \Big|_{u=i/2} \Rightarrow \begin{array}{l} [H^{(n)}, H^{(m)}] = 0 \\ [H^{(n)}, J^A] = 0 \\ [H^{(n)}, Q^A] \sim 0. \end{array}$$

EG.

		γ
STATES	Z^2	0
	$ZW + WZ$	0
	W^2	0

UNPROTECTED STATE $\quad [Z,W][Z,W] \quad \frac{K_6}{2} \cdot 6 \Rightarrow \frac{g_{YM}^2 N}{\pi^2} \frac{3}{4}$

$$\left(u_1 = -u_2 = \frac{1}{\sqrt{12}}\right)$$

$$(P = 0)$$

SUPERCONFORMAL GAUGE THEORY

D=4, $\mathcal{N}=4$ SYM SU(N)

[LARGE N
[PLANAR GRAPHS

't HOOFT 1974

$m=0$ $(\pm 1, 4(\pm \tfrac{1}{2}), 6(0))$ ADJOINT OF SU(N)

$Z = \varphi^1 + i\varphi^2$
$W = \varphi^3 + i\varphi^4$
$Y = \varphi^5 + i\varphi^6$

SO(2,4) × $SU(4)_R$
PSU(2,2|4) SUPERALGEBRA

RADIAL QUANTIZATION ON $R \times S^3$:

- DILATATION GENERATOR IS HAMILTONIAN $D \sim \frac{P^0 + K^0}{2}$
- STATES IN 1↔1 CORRESPONDENCE WITH LOCAL OPERATORS $\mathcal{O}(x)$

$$\lim_{|x|\to 0} \mathcal{O}(x)|0\rangle = |\mathcal{O}\rangle$$

- LOCAL OPERATORS ~ TRACE OF PRODUCT OF LETTERS

↑

ELEMENTARY FIELDS

SINGLE TRACE - LARGE N

$$\mathcal{O}(x) = \Phi^{(1)}_{ij}(x)\,\Phi^{(2)}_{jk}(x)\ldots$$
$$= \mathrm{Tr}\,\Phi^{(1)}(x)\Phi^{(2)}(x)\ldots$$

↑ REPRESENTS A STATE OF A CHAIN OF L SPINS (PARTONS)

$$\Phi^{I,\alpha,\mu}_{ij} \left\{ \begin{array}{l} \Phi^I = \Phi^{IA}(x)\,T^A_{ij} \\ \psi_\alpha = \psi^A_\alpha(x)\,T^A_{ij} \\ F_{\mu\nu} = F^a_{\mu\nu}(x)\,T^A_{ij} \\ D\Phi \ldots \end{array} \right.$$

NON-LOCAL CHARGE AS NOETHER CURRENTS

SYM_4

$$\mathcal{L} = \frac{1}{g^2_{YM}} \mathrm{Tr}\Big(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi^I D^\mu\varphi^I - \tfrac{1}{2}[\varphi^I,\varphi^J][\varphi^I,\varphi^J]$$
$$+ \text{FERMIONS} \Big)$$

EG.

NOETHER CURRENTS FOR $SO(2,4)$:

CLASSICALLY : $j^{\mu A}(x) = K_\nu^{\ A}\, \Theta^{\mu\nu}(x)$

$K_\nu^{\ A}$ CONFORMAL KILLING VECTORS $1 \le A \le 15$

$$\Theta^{\mu\nu} = 2\,\mathrm{Tr}\, F^{\mu\rho}F^\nu_{\ \rho} + 2\,\mathrm{Tr}\, D^\mu\varphi^I D^\nu\varphi^I - g^{\mu\nu}\mathcal{L}$$
$$- \tfrac{1}{3}\mathrm{Tr}\left(D^\mu D^\nu - g^{\mu\nu} D^\rho D_\rho \right)\varphi^I\varphi^I + \text{FERMIONS}$$

$\partial_\mu j^{\mu A}(x) = 0$ FOR ANY $g^2_{YM}N$.

IF WE SET $g^2N = 0$, THEN

THE UNTRACED MATRIX $(\Theta^\mu_{\ \nu})^k_{\ n} = 2F^{\mu\rho}F^\nu_{\ \rho} + 2F^{\nu\rho}F^\mu_{\ \rho}$ + SCALARS + FERMIONS

IS ALSO CONSERVED

AS IS $K_\nu^{\ A}(\Theta^{\mu\nu})^k_{\ n}$

AND

$$\Rightarrow Q^{AB} = \int_M K_\mu^{\ A}(\Theta^{0\mu})^k_{\ n} \int_M K_\nu^{\ B}(\Theta^{0\nu})^n_{\ \ell} \cdots$$

AT $g^2N = 0$.

AT $g^2 N = 0$

$$J^A = \sum_{i=1}^{L} J_i^A$$

$$Q^A = f^A{}_{BC} \sum_{i<j} J_i^B J_j^C$$

$$[J_i^A, J_j^B\} = f^{AB}{}_C J_i^C \delta_{ij}$$

$f^{AB}{}_C$: $PSU(2,2|4) = G$

$Y(G)$: YANGIAN DEFINING RELATIONS

- $[J^A, J^B\} = f^{AB}{}_C J^C$
- $[J^A, Q^B\} = f^{AB}{}_C Q^C$
- $[Q^A, [Q^B, J^C\}\} + [Q^B, [Q^C, J^A\}\} + [Q^C, [Q^A, J^B\}\}$
  $= \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F]$

  $[[Q^A, Q^B\}, [J^C, Q^D\}\} + [[Q^C, Q^D\}, [J^A, Q^B\}\}$
  $= \frac{1}{24} \left(f^{AGL} f^{BEM} f^{KFN} f_{LMN} f_K{}^{CD} + f^{CGL} f^{DEM} f^{KFN} f_{LMN} f_K{}^{AB} \right) \{J_G, J_E, J_F]$

SERRE RELATIONS

J_i^A IN REPRESENTATION R OF G SUCH THAT $R \times \bar{R}$ CONTAINS THE ADJOINT ONLY ONCE.

YANGIAN GENERATORS : $\mathcal{J}_n^A$ WHERE $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, $\mathcal{J}_2^A$ ... FROM COMMUTATORS OF $[Q^A, Q^B\}$...

$n = 0, 1, 2, \ldots, L-1$

WE WILL ASSUME THAT SYM_4 IN PLANAR LIMIT HAS YANGIAN SYMMETRY FOR ALL g^2N.

AS AN EXPANSION IN g^2N

$$\tilde{J}^A = J^A + g^2N\,\delta J^A + O(g^2N)^2$$

$$\tilde{Q}^A = Q^A + g^2N\,\delta Q^A + \text{''}$$

ONE-LOOP CORRECTIONS

ASSUME
$$[\tilde{J}^A, \tilde{J}^B\} = f^{AB}{}_C \tilde{J}^C$$
$$[\tilde{J}^A, \hat{Q}^B\} = f^{AB}{}_C \hat{Q}^C$$

+ SERRE RELATIONS

EXPANDING THESE COMMUTATION RELATIONS TO 1-LOOP

E.G. $[\delta J^A, Q^B] + [J^A, \delta Q^B] = f^{AB}{}_C\, \delta Q^C$

FOR $J^A = D$ DILATATION GENERATOR

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B\, \delta Q^B$$

BARE CONFORMAL DIM.

$\Rightarrow$ $[\delta D, Q^B] = 0$

BARE YANGIAN

1-LOOP HAMILTONIAN SPINCHAIN ; $\boxed{[H, Q^A] = 0 \quad [H, J^A] = 0}$

ALSO $[\delta D, J^B] + [D, \delta J^B] = \lambda^B\, \delta J^B$

$\Rightarrow$ $[\delta D, J^B] = 0$

$[\delta D, \mathcal{A}^B_N] = 0$ δD MUST COMMUTE WITH THE $g^2N = 0$ LIMIT OF WHOLE YANGIAN.

COMMUTATION OF $\mathcal{J}_n^A$ WITH THE PLANAR ONE-LOOP HAMILTONIAN :

$$\delta D \equiv H = \sum_{i=1}^{L-1} H_{i,i+1}$$

A SUM OF OPERATORS ACTING ON NEAREST NEIGHBORS

$$H_{12} = \sum_{j=0}^{\infty} 2h(j)\, P_{12,j} \quad , \qquad h(j) = \sum_{n=1}^{j} \frac{1}{n}$$

ACTS ON A TWO-PARTICLE SYSTEM

HARMONIC NUMBERS

LET V_F BE SPACE OF STATES OF ONE-PARTICLE STATES IN FREE SYM$_4$ ON $\mathbb{R} \times S^3$.

$$V_F \otimes V_F = \oplus \sum_{j=0}^{\infty} V_j$$

DECOMPOSITION IN IRREDUCIBLE REPRESENTATIONS OF $PSU(2,2|4)$

TWO-PARTICLE STATES HAVE :

$$(J_1^A + J_2^A)(J_1^A + J_2^A)\, V_j = j(j+1)\, V_j$$

TWO-PARTICLE SYSTEM :

$$\bigoplus_{j=0}^{\infty} V_j$$

$$\begin{array}{ccccc}
j=0 & j=1 & j=2 & j=3 & \cdots \\
v_0 & v_1 & v_2 & v_3 & \\
P_\mu v_0 & P_\mu v_1 & \vdots & \vdots & \\
P_\mu P_\nu v_0 & \vdots & & & \\
\vdots & & & &
\end{array}$$

$$\updownarrow \ (J_1^A + J_2^A)\, v_j \qquad\qquad Q_{12}^A\, v_j \searrow \sum_j v_j$$

v_j SUPERCONFORMAL PRIMARIES $\quad K_\mu v_j = 0,\ S_\alpha v_j = 0$:

$$v_0 \sim \varphi^I \varphi^J + \varphi^J \varphi^I - \tfrac{1}{3}\delta^{IJ} \varphi^K \varphi^K$$

$$v_1 \sim \varphi^I \varphi^J - \varphi^J \varphi^I$$

$$v_j \sim \sum_{I=1}^{6} \sum_{k=0}^{j-2} \left(c_k^{(j-2)}\, \partial^k \varphi^I\, \partial^{j-2-k} \varphi^I + \ldots \right)$$

$$c_k^{(j-2)} = \frac{(-1)^k}{k!^2\,(j-k-2)!^2}\ , \qquad c_{j-k-2}^{(j-2)} = (-1)^j\, c_k^{(j-2)}$$

WE WILL PROVE

- $(J_1^A - J_2^A)\, V_j \in V_{j-1} \oplus V_{j+1}$.

THEN

$[H_{12}, J_1^A + J_2^A] \,|\lambda(j)\rangle$

FOR ANY $|\lambda(j)\rangle \in V_j$

$= (H_{12} J_{12}^A - J_{12}^A H_{12}) |\lambda(j)\rangle$

$= H_{12} |\rho^A(j)\rangle - J_{12}^A h(j) |\lambda(j)\rangle = (h(j) - h(j)) |\rho^A(j)\rangle = 0.$

$[H_{12}, Q_{12}^A] \,|\lambda(j)\rangle$

$= \frac{1}{4} [H_{12}, [J_{12}^D J_{12}^D, q_{12}^A]] \,|\lambda(j)\rangle$

$= \frac{1}{4} \left(H_{12} J_{12}^2 - j(j+1) H_{12} - 2h(j) J_{12}^2 + 2h(j)\, j(j+1) \right) q_{12}^A |\lambda(j)\rangle$

$= j\,(h(j) - h(j-1))\, |\chi^A(j-1)\rangle$

$\quad + (j+1)\,(h(j+1) - h(j))\, |\rho^A(j+1)\rangle$

$= q_{12}^A \,|\lambda(j)\rangle$

IDENTITY

$Q_{ij}^A = \frac{1}{4} [J_{ij}^2, q_{ij}^A]$

$Q_{ij}^A \equiv f^A{}_{BC} J_i^B J_j^C$

$q_{ij}^A = J_i^A - J_j^A$

$J_{ij}^A = J_i^A + J_j^A$

- $q_{12}^A |\lambda(j)\rangle = |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle$

$h(j+1) - h(j) = \frac{1}{j+1}$

THEN

$[H, J^A] = 0$

$[H, Q^A] = J_1^A - J_L^A$

FOR AN INFINITE CHAIN $L \to \infty$, DROPPING TERMS AT INFINITY, THEN $[H, \mathcal{J}_n^A] = 0$

FOR A FINITE CHAIN WITH PERIODIC BOUNDARY CONDITIONS, WHERE A TOTAL DERIVATIVE WILL SUM TO ZERO $\Rightarrow$ $[H, \mathcal{C}(\mathcal{J}_n^A)] = 0$.

DECOMPOSITION OF $q^A_{12} V_j$:

WE WISH TO SHOW $\quad q^A_{12} V_j \in V_{j-1} \oplus V_{j+1}$

DO THIS BY PROVING :

1) $\quad q^A_{12} V_j \in \bigoplus_{k-j \text{ ODD}} V_k$

2) $\quad q^A_{12} V_j \in \bigoplus_{|k-j| \leq 1} V_k$

PROVE 1) : $\quad q^A_{12} = -q^A_{21} \qquad \left(q^A_{12} = J^A_1 - J^A_2 \right)$

$(j \geq 2)$ $\quad V_j$ HAS HIGHEST WEIGHT STATE v_j

$$v_j \sim \sum_k C_k^{(j-2)} \partial^k \varphi^I \partial^{j-2-k} \varphi^I$$

$$\sim (-1)^j \sum_{k'} C_{k'}^{(j-2)} \partial^{j-2-k'} \varphi^I \partial^{k'} \varphi^I$$

E.G. FOR j EVEN, $q^A_{12} V_j$ IS ODD

UNDER THE EXCHANGE
OF THE TWO SPINS

THEN $\quad q^A_{12} V_j \in \bigoplus_{k \text{ ODD}} V_k$

PROVE 2):

<u>FIRST SHOW</u> $q^A_{12} \, |\psi(j)\rangle \in \bigoplus_{k-j\leq 1} V_k$

FOR $|\psi(j)\rangle$ PRIMARY

$q^A_{12} \, |\psi(j)\rangle \in \bigoplus_{k\leq j+1} V_k$ FOR $j \geq 2$
SINCE $\mathrm{DIM}\ |\psi(j)\rangle = j$
$\mathrm{DIM}\ q^A \leq 1$

$q^A_{12} \, |\psi(1)\rangle \subset \bigoplus_{k\leq 3} V_k$
$\in V_0 \oplus V_2$

FOR $j=1$
$\mathrm{DIM}\ |\psi(1)\rangle = 2$

$q^A_{12} \, |\psi(0)\rangle \subset \bigoplus_{k\leq 3} V_k$
$\subset V_1 \oplus V_3$
$\in V_1$

FOR $j=0$
$\mathrm{DIM}\ |\psi(0)\rangle = 2$

SINCE ONLY STATE IN V_3 WITH $\mathrm{DIM} \leq 3$ IS $|\psi(3)\rangle$ WHICH IS $SU(4)_R$ INVARIANT

BUT $q^A\,|\psi(0)\rangle \sim q^A(\varphi^I\varphi^J + \varphi^J\varphi^I + \cdots)$ IS NOT.

<u>AND SHOW</u> $q^A_{12} \, |\psi(j)\rangle \in \bigoplus_{k-j\geq -1} V_k$

WANT TO SHOW

$\langle \chi | q^A | \psi(j) \rangle = 0$ FOR $k < j-1$ WHEN $|\chi\rangle \in V_k$

$\langle \chi | q^A | \psi(j) \rangle = (\langle \psi(j) | q^{A\dagger} | \chi \rangle)^\dagger$

$\langle \psi(j) | q^{A\dagger} | \chi \rangle = 0$ For $j > k+1$ i.e. $k < j-1$

SINCE IN RADIAL QUANTIZATION $P_\mu^\dagger = K_\mu, \ldots$

$q^{A\dagger}$ IS LINEAR COMBINATION OF q^A

IF $q^A_{12} |\psi(j)\rangle \in \bigoplus_{|k-j| \le 1} V_k$

THEN $q^A_{12} V_j \in \bigoplus_{|k-j| \le 1} V_k$

SINCE $V_j = L_1 L_2 \ldots |\psi(j)\rangle$

$\uparrow$

LOWERING OPERATORS IN $PSU(2,2|4)$

P_μ, Q^a_α

AND $[L, q^A] \sim q^A$

SINCE $[J^B, q^A] = f^{BA}{}_C\, q^C$.

WHY IS $H_{12} = \sum_{j=0}^{\infty} 2h(j)\, P_{12,j}$

THE HAMILTONIAN OF A TWO-SITE INTEGRABLE SPIN MODEL ?

XXX_s MODEL

LOCAL SPIN VARIABLE J_n^A IS IN SPIN S REPRESENTATION OF SL(2)

$H_{n,n+1} = 2\psi(J+1)$ INTEGRABLE HAMILTONIAN

WHERE $(J_n^A + J_{n+1}^A)(J_n^A + J_{n+1}^A) = J(J+1)$

$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = -\gamma + h(x-1)$ DIGAMMA FUNCTION

$H_{n,n+1} = -2\gamma + 2h(J)$

R-MATRIX

$$R_{12}(u) = \frac{\Gamma(J+1-iu)}{\Gamma(J+1+iu)} P_{12}$$

P_{12} IS PERMUTATION $\quad P_{12} J_1^A P_{12} = J_2^A$

$$H_{n,n+1} = i\frac{d}{du}\left(\frac{\Gamma(J+1-iu)}{\Gamma(J+1+iu)}\right)\Bigg|_{u=0} = 2\psi(J+1)$$

$[H, J^A] = 0$

$[H, Q^A] = q_{1L}^A$

$$H_{12} = 2\psi(J+1) = 2f(J_1^A J_2^A)$$

$$J(J+1) = (J_1^A + J_2^A)(J_1^A + J_2^A) = J_1^A J_1^A + J_2^A J_2^A + 2J_1^A J_2^A$$
$$= 2s(s+1) + 2J_1^A J_2^A$$

$$x \equiv J_1^A J_2^A = \tfrac{1}{2}\left(\ell(\ell+1) - 2s(s+1)\right)$$

WRITE $\psi(J+1)$ AS A POLYNOMIAL IN $J_1^A J_2^A$:

$$\psi(\ell+1) = -\gamma + \sum_{n=1}^{\ell} \frac{1}{n}$$ VALUE OF FUNCTION IS KNOWN FOR INTEGER ℓ.

$$\psi(J+1) = f(x) = \sum_{j=0}^{2s} \psi(j+1) \prod_{\substack{\ell=0 \\ \ell\neq j}}^{2s} \frac{x - x_\ell}{x_j - x_\ell}$$

EG. $\boxed{s=\tfrac{1}{2}}$ $\quad x_\ell = \tfrac{1}{2}\left(\ell(\ell+1) - 3/2\right) \Rightarrow x_0 = -3/4,\ x_1 = \tfrac{1}{4}$

$$\psi(J+1) = f(x) = \psi(1)\frac{x-x_1}{x_0-x_1} + \psi(2)\frac{x-x_0}{x_1-x_0}$$

$f(x_0) = \psi(1)$, $\quad \psi(1) = -\gamma$

$f(x_1) = \psi(2)$, $\quad \psi(2) = -\gamma + 1$

$$\Rightarrow \quad f(x) = \left(\psi(2) - \psi(1)\right)x + \text{CONSTANT} = x + \text{CONST.}$$

$$\boxed{\psi(J+1) = J_1^A J_2^A + \text{CONST.}}$$

$\boxed{s=1}$ $\quad x_\ell = 1/2\left(\ell(\ell+1) - 4\right) \Rightarrow$ $x_0 = -2$, $x_1 = -1$, $x_2 = 1$

$$f(x) = \psi(1)\frac{x-x_1}{x_0-x_1}\frac{x-x_2}{x_0-x_2} + \psi(2)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + \psi(3)\frac{x-x_0}{x_2-x_0}\frac{x-x_1}{x_2-x_1}$$

$$= \tfrac{1}{4}(x - x^2) + \text{CONST.}$$

$$\Rightarrow \quad \psi(J+1) = \tfrac{1}{4}\left(J_1^A J_2^A - (J_1^A J_2^A)^2\right) + \text{CONST.}$$

NON-LOCAL SYMMETRIES OF σ-MODELS

$g_{ab}(\sigma,\tau) = e^{x^\mu(\sigma,\tau) T^\mu_{ab}} \in$ GROUP G (TARGET SPACE)

$S = \int d^2\sigma \; \mathrm{Tr}\left(g^{-1}\partial_\mu g \; g^{-1}\partial_\mu g\right) + \ldots \sim S_{\text{WORLDSHEET}}$

$\mu = 0, 1.$

$J^\mu(\sigma,\tau) \equiv g^{-1}\partial^\mu g \in$ LIE ALGEBRA OF G

EQ. OF MOTION $\partial_\mu J^\mu = 0$

FLAT CONNECTION $\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0$

A FAMILY OF FLAT CONNECTIONS

$a_\mu = \alpha J_\mu + \beta \varepsilon_{\mu\nu} J^\nu$

$\partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = 0$

IMPLIES $\alpha^2 - \alpha - \beta^2 = 0.$

$a_\mu^{\rho\pm} \Longleftarrow$ LET $\alpha = \frac{1}{2}(1 \pm \cosh\rho)$

$\beta = \frac{1}{2}\sinh\rho$

INFINITE SET OF CONSERVED CHARGES:

$$Q^{\rho\pm} = P\, e^{-\int_{-\infty,t}^{\infty,t} ds^\mu a_\mu} = \mathcal{U}(\infty,t;-\infty,t)$$

$$\partial_0 Q^{\rho\pm} = \int_{-\infty}^{\infty} dx \; \mathcal{U}(\infty,t;x,t)\, \underbrace{\dot{a}_1(x,t)}_{= \partial_1 a_0 - [a_0, a_1]}\, \mathcal{U}(x,t;-\infty,t) = 0$$

$$Q^{\rho\pm} = 1 + \sum_{n=0}^{\infty} \rho^n \mathcal{J}_n \,, \quad \mathcal{J}_n^A = (\mathcal{J}_n)_{ab} T^A_{ab}$$

$$a_\mu^{\rho\pm} = \frac{1}{2}\rho\,\varepsilon_{\mu\nu} J^\nu \pm \frac{1}{4}\rho^2 J_\mu + O(\rho^3)$$

$$\mathcal{J}_0 = \int_{-\infty}^{\infty} dx\, J_0 \longrightarrow J^A$$

$$\mathcal{J}_1 = -\frac{1}{4}\int_{-\infty}^{\infty} dx\, J_1 + \frac{1}{4}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\, J_0(x,t)\, J_0(y,t)$$

$$\longrightarrow Q^A$$

NON-LOCAL SYMMETRIES OF TYPE IIB STRING ON $AdS_5 \times S^5$

NEGLECTING FOR THE MOMENT CLOSED STRING BOUNDARY CONDITIONS, THE NON-LOCAL CHARGES FORM THE YANGIAN ALGEBRA $Y(PSU(2,2|4))$.

AdS/CFT CORRESPONDENCE

D=4 N=4 SU(N) YANG MILLS — IIB SUPERSTRING ON $AdS_5 \times S^5$

LARGE N $\longleftrightarrow$ STRING TREE GRAPHS

LARGE g^2N — SG LIMIT $\alpha' \to 0$
BENA, POLCHINSKI, ROIBAN

SMALL g^2N
DNW
MINAHAN, ZAREMBO
BEISERT, STAUDACHER, KRISTJANSEN

$Y(PSU(2,2|4))$ FOR ALL g^2N.

HIGHER LOOP INTEGRABILITY

$$[\tilde{J}^A, \tilde{J}^B] = f^{AB}{}_C \tilde{J}^C \qquad \tilde{J}^A(g^2N)$$

$$[\tilde{J}^A, \tilde{Q}^B] = f^{AB}{}_C \tilde{Q}^C \qquad \tilde{Q}^A(g^2N)$$

THEN $[\tilde{D}, \tilde{J}^B] = \lambda^B \tilde{J}^B$ $\qquad \tilde{D} = D + g^2N\,\delta D + O(g^2N)^2$

$[\tilde{D}, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$ $\qquad = D + \Delta D(g^2N)$

ALSO $[D, \tilde{J}^B] = \lambda^B \tilde{J}^B$

$[D, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$

SO FOR THE EXACT ANOMALOUS DIMENSION OPERATOR $\Delta D = \tilde{D} - D$ FOR LARGE N

COMMUTES WITH THE

EXACT YANGIAN (IN THE PLANAR LIMIT).

$$[\underset{\text{EXACT}}{\Delta D}, \underset{\text{EXACT}}{\tilde{J}^B}] = 0 \qquad \text{c.f.} \qquad [\underset{\text{ONE-LOOP}}{\delta D}, \underset{\text{BARE}}{J^B}] = 0$$

$$[\Delta D, \tilde{Q}^B] = 0 \qquad\qquad [\delta D, Q^B] \sim 0$$