Homework 8
Classical Mechanics

Due Nov 25

Name Justin Moore

****From the book, chapter 10, solve 4 of the following problems (earn extra grade for solving all problems):

10.6, 10.7, 10.8, 10.13, 10.17

CLASSICAL MECHANICS HOMEWORK # 8

CH. 10
6, 7, 8, 13, 17

10.6 Charged particle constrained to move in plane under influence of central force potential $V = \frac{1}{2}kr^2$ AND CONSTANT $\vec{B}$

$$\vec{A} = \frac{1}{2}\vec{B} \otimes \vec{r}$$

- Set up H-J EQ. FOR HAMILTON'S CHARACTERISTIC FUNCTION IN PLANE POLAR COORDINATES.
- SEPARATE AND REDUCE TO QUADRATURES
- DISCUSS MOTION IF $P_\theta = 0$ AT $t=0$ ← what is T.F... gross

FIRST WE NEED TO BUILD THE HAMILTONIAN FOR USE IN THE H J EQ PROCEDURE.

START W/ LAGRANGIAN.

2-D POLAR

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \underbrace{q\vec{A}\cdot\vec{v}} - \frac{1}{2}kr^2$$

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}Br(\hat{z}\times\hat{r}) = \frac{1}{2}B\hat{\phi}$$

$$q\frac{1}{2}Br\hat{\phi}\cdot(r\dot{\phi}\hat{\phi}) = \frac{q}{2}Br^2\dot{\phi}$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{q}{2}Br^2\dot{\phi} - \frac{1}{2}kr^2$$

TAKE MOMENTA FROM LAGRANGIAN VIA (8.2)

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = P_r/m$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + \frac{qBr^2}{2} \longrightarrow \dot{\phi} = \frac{1}{mr^2}\left(P_\phi - \frac{qBr^2}{2}\right)$$

$$\dot{\phi} = \frac{P_\phi}{mr^2} - \frac{qB}{2m}$$

$$H = p_i\dot{q}_i - L$$

$$H = \frac{P_r^2}{m} + \frac{P_\phi^2}{mr^2} - \frac{P_\phi qB}{2m} - \frac{m}{2}\left(\frac{P_r^2}{m^2}\right) + \frac{m}{2}r^2\left(\frac{P_\phi}{mr^2} - \frac{qB}{2m}\right)^2 + \frac{q}{2}Br^2\left(\frac{P_\phi}{mr^2} - \frac{qB}{2m}\right) + \frac{1}{2}kr^2$$

$$H = \frac{P_r^2}{m} + \frac{P_\phi^2}{mr^2} - \frac{P_\phi qB}{2m} - \frac{mP_r^2}{2m^2} - \frac{m}{2}r^2\left(\frac{P_\phi^2}{m^2r^4} - \frac{2P_\phi qB}{2m^2r^2} + \frac{q^2B^2}{4m^2}\right) + \frac{q}{2}B\frac{P_\phi}{m} - \frac{q^2B^2r^2}{4m} + \frac{1}{2}kr^2$$

$$\frac{P_\phi^2}{2mr^2} + \frac{P_\phi qB}{2m} + \frac{q^2B^2r^2}{8m}$$

$$H = \frac{P_r^2}{m} + \frac{P_\phi^2}{mr^2} - \cancel{\frac{P_\phi qB}{2m}} - \frac{mP_r^2}{2m^2} + \frac{P_\phi^2}{m^2r^2} - \frac{P_\phi qB}{2m} + \cancel{\frac{qBP_\phi}{2m}} + \frac{1}{2}kr^2$$

HAMILTONIAN

$$H = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} + \frac{q^2B^2r^2}{8m} + \frac{1}{2}kr^2 - \frac{qBP_\phi}{2m}$$

$$H + \frac{\partial S}{\partial t} = 0 \qquad S = W_r + W_\phi - \alpha t \quad \text{because no explicit } t \text{ dependence}$$

10.72 $W_\phi = \phi\alpha_\phi$

Also, no ϕ so $S = W_r + \phi\alpha_\phi - \alpha t$

(10.7) $p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i}$ $S = W_r + \phi\,\alpha_\phi - \alpha t$

Rewriting H for Convenience,

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{q^2B^2r^2}{8m} + \frac{1}{2}kr^2 - \frac{qBp_\phi}{2m}$$

(10.32) $$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\phi^2}{2mr^2} + \frac{q^2B^2r^2}{8m} + \frac{1}{2}kr^2 - \frac{qB\alpha_\phi}{2m} = \alpha$$

NOW SOLVE FOR S VIA W,

$$W = \underbrace{\int\left(2m\alpha - \frac{\alpha_\phi^2}{r^2} - \frac{q^2B^2r^2}{4} - mkr^2 + qB\alpha_\phi\right)^{1/2} dr}_{\text{CALL THIS BEAST } \tilde{\chi}^{(1/2)}}$$

$$S = \tilde{\chi}^{(1/2)} + \phi\,\alpha_\phi - \alpha t$$

So, we solve for the constants

$$\boxed{\begin{aligned} \beta &= \frac{\partial S}{\partial \alpha} = \cancel{2}m\frac{1}{\cancel{2}}\tilde{\chi}^{(-1/2)} - t \\ \beta_\phi &= \frac{\partial S}{\partial \alpha_\phi} = \int\left(qB - \frac{2\alpha_\phi}{r^2}\right)\tilde{\chi}^{(-1/2)} + \phi - t \end{aligned}}$$

THIS IS QUADRATURES

When $p_\phi = 0$ when $t=0$, PLUG IN, $t=0$, $\alpha_\phi = 0$

$$\boxed{\begin{aligned} \beta &= \cancel{2}m\int\left(2m\alpha - mkr^2 - \frac{q^2B^2r^2}{4}\right)^{1/2} dr + t \quad \leftarrow t=0 \\ \beta_\phi &= qB\int\left(2m\alpha - mkr^2 - \frac{q^2B^2r^2}{4}\right)^{1/2} dr + \phi - t \end{aligned}}$$

SIMPLE HARMONIC MOTION

JUST FOR KICKS I DID THE W INTEGRAL IN MATHEMATICA...
THAT SHIT IS GROSS!

J. Moery
p. 3

$$\int \sqrt{2\,m\,\alpha - \frac{\alpha\phi^2}{r^2} - \frac{q^2\,B^2\,r^2}{4} - m\,k\,r^2 + q\,B\,\alpha\phi}\;dr$$

$$W \quad \Big(r\sqrt{-4\,k\,m\,r^2 - B^2\,q^2\,r^2 + 8\,m\,\alpha + 4\,B\,q\,\alpha\phi - \frac{4\,\alpha\phi^2}{r^2}} \Big(2\sqrt{4\,k\,m + B^2\,q^2}\;\alpha\phi\,\mathrm{Log}\big[r^2\big] - 2\,(2\,m\,\alpha + B\,q\,\alpha\phi)\,\mathrm{Log}\Big[4\,k\,m\,r^2 + B^2\,q^2\,r^2 - 4\,m\,\alpha - 2\,B\,q\,\alpha\phi + \sqrt{4\,k\,m + B^2\,q^2}\;\sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2}\Big] + \sqrt{4\,k\,m + B^2\,q^2}\,\Big(\sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2} - 2\,\alpha\phi\,\mathrm{Log}\Big[2\,m\,r^2\,\alpha - \alpha\phi\,\Big(-B\,q\,r^2 + 2\,\alpha\phi + \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2}\Big)\Big]\Big)\Big)\Big) \Big/ \Big(4\sqrt{4\,k\,m + B^2\,q^2}\;\sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2}\Big)$$

$$S = \int \sqrt{2\,m\,\alpha - \frac{\alpha\phi^2}{r^2} - \frac{q^2\,B^2\,r^2}{4} - m\,k\,r^2 + q\,B\,\alpha\phi}\;dr - \phi\,\alpha\phi - \alpha\,t;$$

```
D[S, α] // FullSimplify
D[S, αϕ] // FullSimplify
```

J. Moore
p. 4

$$
\begin{aligned}
\Bigg(& 4\,m^2\,r\,\alpha^2 \sqrt{-4\,k\,m\,r^2 + 8\,m\,\alpha - \frac{\left(B\,q\,r^2 - 2\,\alpha\phi\right)^2}{r^2}} + \\
& 2\,m\,\alpha \Bigg(\left(2\,B\,q + \sqrt{4\,k\,m + B^2\,q^2}\right) r \sqrt{-4\,k\,m\,r^2 + 8\,m\,\alpha - \frac{\left(B\,q\,r^2 - 2\,\alpha\phi\right)^2}{r^2}}\ \alpha\phi + \\
& \quad 2 \sqrt{4\,k\,m + B^2\,q^2}\ t\,\alpha \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2} \Bigg) + \\
& \alpha\phi \Bigg(B^2\,q^2\,r \sqrt{-4\,k\,m\,r^2 + 8\,m\,\alpha - \frac{\left(B\,q\,r^2 - 2\,\alpha\phi\right)^2}{r^2}}\ \alpha\phi - \\
& \quad 4\,k \sqrt{4\,k\,m + B^2\,q^2}\ t\,\alpha\phi \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2} + \\
& \quad B\,q \sqrt{4\,k\,m + B^2\,q^2} \Bigg(r \sqrt{-4\,k\,m\,r^2 + 8\,m\,\alpha - \frac{\left(B\,q\,r^2 - 2\,\alpha\phi\right)^2}{r^2}}\ \alpha\phi + \\
& \qquad 4\,t\,\alpha \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2} \Bigg) \Bigg) + \\
& 4\,m\,r \sqrt{-4\,k\,m\,r^2 + 8\,m\,\alpha - \frac{\left(B\,q\,r^2 - 2\,\alpha\phi\right)^2}{r^2}}\ \left(m\,\alpha^2 + \alpha\phi\ (B\,q\,\alpha - k\,\alpha\phi)\right) \mathrm{Log}\Big[\\
& \quad 4\,k\,m\,r^2 - 4\,m\,\alpha + B\,q \left(B\,q\,r^2 - 2\,\alpha\phi\right) + \sqrt{4\,k\,m + B^2\,q^2}\ \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2}\ \Big] \Bigg) \Big/ \\
\Bigg(& 4 \sqrt{4\,k\,m + B^2\,q^2}\ \sqrt{4\,k\,m\,r^4 + B^2\,q^2\,r^4 - 8\,m\,r^2\,\alpha - 4\,B\,q\,r^2\,\alpha\phi + 4\,\alpha\phi^2} \\
& \left(-m\,\alpha^2 + \alpha\phi\ (-B\,q\,\alpha + k\,\alpha\phi)\right) \Bigg)
\end{aligned}
$$

$$\Rightarrow \quad \frac{\partial J}{\partial \alpha} = \beta$$

$$\left(\frac{r\sqrt{-4\,k\,m\,r^2+8\,m\,\alpha-\frac{(B\,q\,r^2-2\,\alpha\phi)^2}{r^2}}\;(B\,q\,\alpha-2\,k\,\alpha\phi)\left(2\,m\,\alpha+\left(B\,q+\sqrt{4\,k\,m+B^2\,q^2}\right)\alpha\phi\right)}{\sqrt{4\,k\,m+B^2\,q^2}\;\left(-m\,\alpha^2+\alpha\phi\,(-B\,q\,\alpha+k\,\alpha\phi)\right)}-\right.$$

$$4\sqrt{4\,k\,m\,r^4+B^2\,q^2\,r^4-8\,m\,r^2\,\alpha-4\,B\,q\,r^2\,\alpha\phi+4\,\alpha\phi^2}\;\phi+\frac{1}{\sqrt{4\,k\,m+B^2\,q^2}}$$

$$2\,r\sqrt{-4\,k\,m\,r^2+8\,m\,\alpha-\frac{(B\,q\,r^2-2\,\alpha\phi)^2}{r^2}}\;\Big(-B\,q\,\mathrm{Log}\Big[4\,k\,m\,r^2-4\,m\,\alpha+B\,q\,(B\,q\,r^2-2\,\alpha\phi)+\sqrt{4\,k\,m+B^2\,q^2}$$

$$\sqrt{4\,k\,m\,r^4+B^2\,q^2\,r^4-8\,m\,r^2\,\alpha-4\,B\,q\,r^2\,\alpha\phi+4\,\alpha\phi^2}\Big]+\sqrt{4\,k\,m+B^2\,q^2}\;\Big(\mathrm{Log}[-i\,r]+\mathrm{Log}[i\,r]-$$

$$\left.\mathrm{Log}\Big[2\,m\,r^2\,\alpha-\alpha\phi\Big(-B\,q\,r^2+2\,\alpha\phi+\sqrt{4\,k\,m\,r^4+B^2\,q^2\,r^4-8\,m\,r^2\,\alpha-4\,B\,q\,r^2\,\alpha\phi+4\,\alpha\phi^2}\Big)\Big]\Big)\Big)\right)\Big/$$

$$\left(4\sqrt{4\,k\,m\,r^4+B^2\,q^2\,r^4-8\,m\,r^2\,\alpha-4\,B\,q\,r^2\,\alpha\phi+4\,\alpha\phi^2}\right)$$

$$\frac{\partial J}{\partial \alpha_\phi}=\beta_\phi$$

(10.8) Suppose potential in 1-D.o.F. problem is linearly dependent upon time

$$H = \frac{p^2}{2m} - mAtx$$

$A =$ const.

Solve dynamics of Hamilton's principle function
initial conditions $\Rightarrow$ $t=0,\ x=0,\ p=mv_0$

(10.23) H-J Form

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mAtx + \frac{\partial S}{\partial t} = 0$$

Guess form of solution S as two time dependent functions,

$$S = \tilde{A}(t)x + \tilde{B}(t)$$

$$H + \frac{\partial S}{\partial t} = 0 \Rightarrow H + \tilde{A}'(t)x + \tilde{B}'(t) = 0$$

$$\underbrace{\frac{1}{2m}\tilde{A}(t)^2}_{x^0} \underbrace{-mAtx + \tilde{A}'(t)x}_{\text{need to sum to zero}} + \underbrace{\tilde{B}'(t)}_{x^0} = 0$$

$x^1 \quad x^1$

$$mAt = \tilde{A}'(t) \qquad \tilde{A}(t) = \tfrac{1}{2}mAt^2 + C_0 \text{ by integration.}$$

$$\tilde{B}'(t) = -\frac{1}{2m}\tilde{A}(t)^2$$

$$= -\frac{1}{2m}\left(\frac{m}{2}At^2 + C_0\right)^2$$

$$= -\frac{1}{2m}\left(\frac{m^2}{4}A^2t^4 + 2C_0\frac{m}{2}At^2 + C_0^2\right)$$

C_1 $\qquad -\frac{1}{2m}C_1 = C_2$

$$\tilde{B}'(t) = -\frac{1}{8}A^2t^4 + \frac{-1}{2}C_0At^2 + C_2$$

By Integration, $\tilde{B}(t) = -\frac{1}{8}A^2\frac{1}{5}t^5 - \frac{1}{2}AC_0\frac{1}{3}t^3 + C_2t + C_3$

$$\tilde{B}(t) = -\frac{1}{40}A^2t^5 - \frac{1}{6}AC_0t^3 + C_2t + C_3$$

$$\tilde{A}(t) = \frac{1}{2}mAt^2 + C_0$$

$$S = \frac{1}{2}mAt^2x + C_0x - \frac{A^2t^5}{40} - \frac{AC_0t^3}{6} + C_2t + \cancel{C_3}$$

Free, we can set to 0, because taking derivatives won't include this term

$S = \frac{1}{2} m A t^2 x + C_0 x - \frac{A^2 t^5}{40} - \frac{A C_0 t^3}{6} + C_2 t$

$C_2 = \frac{-1}{2m} C_1$ AND $C_1 = C_0^2$

$C_2 = -\frac{C_0^2}{2m}$

J. Moore p. 7

$$\frac{\partial S}{\partial C_0} = \beta = x - \frac{A t^3}{6} - \frac{C_0}{m} t$$

$$x = \beta + \frac{A}{6} t^3 + \frac{C_0}{m} t$$

$$x(t=0) = 0 \xRightarrow{\text{IMPLIES}} \beta = 0$$

$$\dot{x}(t=0) = v_0 \xRightarrow{\text{IMPLIES}} \dot{x} = \frac{A t^2}{2} + \frac{C_0}{m} \cancel{v_0}$$

$$C_0 = \left(v_0 - \frac{A}{2} t^2\right) m$$
$$= m v_0 - \frac{A m}{2} t^2$$

Therefore, plug back in...

OTHERS GOT $\rightarrow x = v_0 t + \frac{A}{6} t^3$? I think C_0 is solved for wrong...

$$x = \frac{A}{6} t^3 + v_0 t - \frac{A}{2} t^3$$

$$x = v_0 t - \frac{2A}{6} t^3$$

$\frac{3A}{6}$

$$\boxed{x = v_0 t - \frac{1}{3} A t^3}$$

(10.13) PARTICLE MOVES IN PERIODIC MOTION IN 1-D UNDER INFLUENCE OF A POTENTIAL $V(x) = F|x|$, WHERE F IS A CONSTANT

USE ACTION ANGLE VARIABLES METHOD TO FIND THE PERIOD OF MOTION AS A FUNCTION OF THE PARTICLE'S ENERGY.

$V(x)$

$$H = E = T + V$$
$$H = E = \frac{p_x^2}{2m} + F|x|$$

Need P in function of F and E

$$p_x = \pm\sqrt{2m(E - F|x|)}$$

Now find J in terms of $p(q, E)$

$$J = \oint p(q, E)\,dq$$ IN OUR CASE, $q = |x|$, because F is just a constant

$$J = \oint \sqrt{2m(E - F|x|)}\,dx$$

$\frac{E}{F} = x = 0$

So, LOOKING AT NEXT PAGE WE CAN SEE A GENERAL DEPICTION OF THE CURVE TO INTEGRATE

PLOTTED IN MATHEMATICA ON NEXT PAGE

WE CAN INTEGRATE $\int_0^{E/F} \left(2m(E - F|x|)\right)^{1/2} dx$

$$= \int_0^{E/F} (2mF|x|)^{1/2}\,dx$$

$$= \frac{1}{\cancel{2}mF}\,\frac{2}{3}\left(2mF|x|\right)^{3/2}\Big|_0^{E/F}$$

$$= \frac{1}{3mF}\left(2mF(E/F - 0)\right)^{3/2}$$

$$= \frac{1}{3mF}(2mE)^{3/2}$$

FOR THE COMPLETE LOOP, WE HAVE 4 * THIS ANSWER,

THEREFORE $J = \frac{4}{3mF}(2mE)^{3/2}$

The inverse of the Period $\frac{1}{\tau} = \frac{\partial E}{\partial J}$ where E is solved for in terms of J, but re-arranging, we have $\tau = \frac{\partial J}{\partial E}$

Therefore

$$\tau = \frac{\partial J}{\partial E} = \frac{2}{mF}(2mE)^{1/2} \cdot 2m \Rightarrow \frac{4\cancel{m}\sqrt{2mE}}{\cancel{m}F}$$

$$\boxed{\tau = \frac{4\sqrt{2mE}}{F}}$$

J. Moore
p. 9

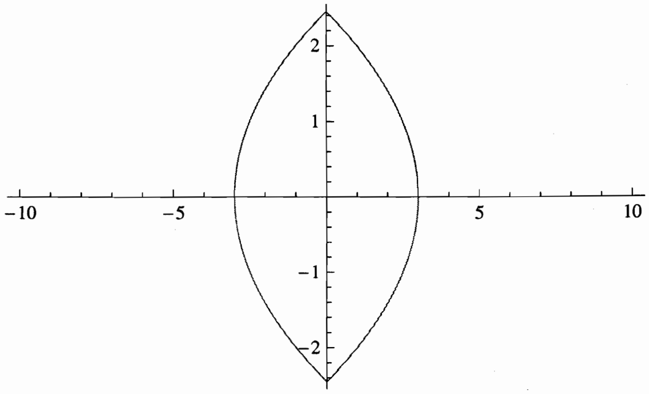
Justin Moore
Mechanics Prob 10.13

Showing the loop integral for the J integral from $\oint P(|x|, E)\, d|x|$

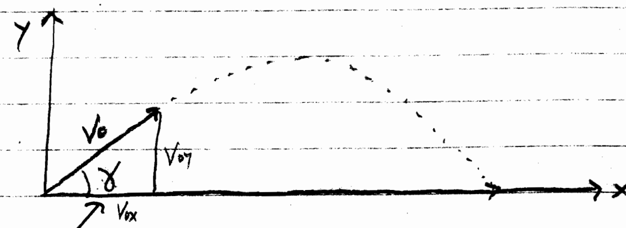
Random parameters inserted to see the shape of the integral.

```
In[1]:= E1 = 3; F1 = 1; m = 1;
        Plot[{√(2 m (E1 - F1 Abs[x])), -√(2 m (E1 - F1 Abs[x]))}, {x, -10, 10}]
```

Out[2]=



(10.17) Solve the problem of point projectile in vertical plane using Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t=0$ from the origin with velocity V_0, making angle α with the horizontal.



CHANGE TO γ B/C HAMILTONIAN JACOBI HAS LOTS OF α

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy = E$$ // NO TIME DEPENDENCE

USING (10.7) $\quad p_i = \dfrac{\partial S(q, \alpha, t)}{\partial q_i}$

$$p_x = \pm\sqrt{2mE - p_y^2 - 2m^2gy}$$

$$p_x = \pm\sqrt{2mE - p_x^2 - 2m^2gy}$$

$$p_x = \frac{\partial S}{\partial x}, \quad p_y = \frac{\partial S}{\partial y}$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + 2m^2gy\right] + \frac{\partial S}{\partial t} = 0 \qquad (10.20)$$

WE HAVE NO EXPLICIT TIME, SO WE CAN USE (10.14) TO REPLACE A TIME INDEPENDENT S WITH W, NAMELY;

STILL TRYING TO FIGURE OUT WHAT I'M DOING...

$$S(q, \alpha, t) = W(q, \alpha) - at$$

FOR OUR SITUATION, WE HAVE TWO q's : $x + y$

$$S(x, y, \alpha_x, \alpha, t) = W_x(x, \alpha_x) + W_y(y, \alpha_y) - \alpha t \qquad (10.31)$$

CONTINUE ON BACK --

No Explicit time dependence, so $S = W_x + W_y - \alpha t$
but $W_x = q_i \alpha_i = x\alpha_x$ because x does not appear in H
So, $S = x\alpha_x + W_y - \alpha t$

OVER

J. Moore
p. 11

HAMILTON JACOBI EQ ASSUMES THE FORM:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+2m^2yg\right]=\alpha$$

$$\boxed{\frac{1}{2m}\left[\left(\frac{\partial W}{\partial y}\right)^2+2m^2gy\right]=\alpha_y}$$

AND

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2+\alpha_y=\alpha \qquad \alpha-\alpha_y=\alpha_x$$

$$\alpha=\alpha_x+\alpha_y$$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2=\alpha-\alpha_y$$

$$\boxed{\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2=\alpha_x}$$

$$\alpha_x=\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 \Rightarrow \frac{\partial W}{\partial x}=(2m\alpha_x)^{1/2}=p_x$$

still...

$$\alpha_y=\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2+mgy \Rightarrow \frac{\partial W}{\partial y}=\left[2m(\alpha_y-mgy)\right]^{1/2}=p_y$$

$$W_y=\int\sqrt{2m\alpha_y-2m^2gy}\,dy=\frac{1}{-2m^2g}\,\frac{2}{3}(2m\alpha_y-2m^2gy)^{3/2}=-\frac{1}{3m^2g}(2m\alpha_y-2m^2gy)^{3/2}$$

For $S_y=W_y-\alpha_y t$,

$$S_y=\int\sqrt{2m\alpha_y-2m^2gy}\,dy-\alpha_y t=-\frac{1}{3m^2g}(2m\alpha_y-2m^2gy)^{3/2}-\alpha t$$

$$S_x=W_x-\alpha_x t=\sqrt{2m\alpha_x}\,x-\alpha_x t$$

W_x is just gixi because x does not appear in the Hamiltonian, cyclic.

$$\boxed{S=x\alpha_x-\frac{1}{3m^2g}(2m\alpha-2m^2gy-\alpha_x^2)^{3/2}-\alpha t}$$

(2) $$\beta=\frac{\partial S}{\partial\alpha}=-\frac{1}{mg}(2m\alpha-2m^2gy-\alpha_x^2)^{1/2}-t$$

$$\beta_x=\frac{\partial S}{\partial\alpha_x}=x+\frac{1}{\cancel{2}m^2g}(-\cancel{2}\alpha_x)(2m\alpha-2m^2gy-\alpha_x^2)^{1/2}$$

$$\beta_x=x+\frac{\alpha_x}{m^2g}\underbrace{(2m\alpha-2m^2gy-\alpha_x^2)^{1/2}}_{=-(\beta+t)mg}$$

$$\beta_x=x-\frac{\alpha_x}{m^{\cancel{2}}g}(\cancel{mg}(\beta+t))$$

$$=x-\frac{\alpha_x}{m}(\beta+t)$$

$$\boxed{x=\beta_x+\frac{\alpha_x}{m}(\beta+t)}$$

From (2) $$(\beta+t)^2(mg)^2-2m\alpha+\alpha_x^2=-2m^2gy$$

$$\boxed{y=-\frac{g}{2}(\beta+t)^2+\frac{\alpha}{mg}-\frac{\alpha_x^2}{2m^2g}}$$

$\boxed{5.17 \text{ cont...}}$

Now we need to solve for our four constants, $\beta, \beta_x, \alpha, \alpha_x$ in terms of the initial conditions set by the problem

$t=t_0;\quad x=0 \qquad y=0$

$t=t_0;\quad \dot{x}=V_0\cos\gamma \qquad \dot{y}=V_0\sin\gamma$

$$0=\beta x+\frac{\alpha_x}{m}\beta \Rightarrow \boxed{\beta_x=-\frac{\alpha_x}{m}\beta} \Rightarrow \beta_x=-\frac{V_0\cos\gamma m}{m}\cdot -\frac{V_0}{g}\sin\gamma$$

$$\boxed{\beta_x=\frac{V_0^2}{g}\cos\gamma\sin\gamma}$$

$$0=-\frac{g}{2}\beta^2+\frac{\alpha}{mg}-\frac{\alpha_x^2}{2m^2g}$$

$$\dot{x}=\frac{\alpha_x}{m}=V_0\cos\gamma \Rightarrow \boxed{\alpha_x=V_0\cos\gamma m}$$

$$\dot{y}=-g(\beta+t)\Big|_{t=0}=V_0\sin\gamma \qquad \boxed{\beta=-\frac{V_0}{g}\sin\gamma}$$

$$0=-\frac{g}{2}\left(\frac{V_0^2}{g^2}\sin^2\gamma\right)+\frac{\alpha}{mg}-\frac{V_0^2\cos\gamma^2 m^2}{2m^2g}$$

$$-\frac{V_0^2}{2g}\left(\cos^2\gamma+\sin^2\gamma\right)+\frac{\alpha}{mg}$$

$$\boxed{\alpha=\frac{mV_0^2}{2}}$$

NOW WE HAVE ALL CONSTANTS TO PLUG BACK INTO OUR X AND Y EQ'S

$$X=\frac{V_0^2}{g}\cos\gamma\sin\gamma+\frac{V_0\cos\gamma \cancel{m}}{\cancel{m}}\left(-\frac{V_0}{g}\sin\gamma+t\right)$$

$$-\left(\cancel{\frac{V_0^2}{g}\cos\gamma\sin\gamma}-V_0\cos\gamma t\right)$$

$$\boxed{X=V_0\cos\gamma t}$$

$$y=-\frac{g}{2}\left(-\frac{V_0}{g}\sin\gamma+t\right)^2+\frac{mV_0^2}{2mg}-\frac{V_0^2\cos^2\gamma m^2}{2m^2g}$$

$$=-\frac{g}{2}\left(\frac{V_0^2}{g^2}\sin^2\gamma+t^2-2\frac{V_0}{g}\sin\gamma t\right)+\frac{V_0^2}{2g}-\frac{V_0^2\cos^2\gamma}{2g}$$

$$=-\frac{1}{2}gt^2-\frac{V_0^2}{2g}\sin^2\gamma+g\frac{V_0}{g}\sin\gamma t+\frac{V_0^2}{2g}-\frac{V_0^2\cos^2\gamma}{2g}$$

$$=-\frac{1}{2}gt^2+V_0\sin\gamma t\quad \cancel{\frac{V_0^2}{2g}}-\cancel{\frac{V_0^2}{2g}\left[\sin^2\gamma+\cos^2\gamma\right]}$$

COOL, BUT LENGTHY!

$$\boxed{y=-\frac{1}{2}gt^2+V_0\sin\gamma t}$$

These are exactly what we would expect in Newtonian!