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Homework #8

Q.1.

Let the density of states of the electrons in some sample be assumed to be a constant D, $g(\varepsilon) = D$, for energy $\varepsilon > 0$ and zero for $\varepsilon < 0$. The total number of electrons is N.

(1) Calculate the Fermi energy.

(2) Show that the chemical potential μ for degenerate electrons (low T approximation) is independent of temperature.

(3) What is relationship between pV and U?

(4) Show that the specific heat at low T is given by $C_V = \frac{1}{3}\pi^2 k_B^2 DT$.

Start out with these two equations, one is for the grand partition function for given (T, V, z) and the other is for determining z for given (T, V, N). From these two equations, statistical and thermodynamic quantities can be obtained for given (T, V, N).

$$\frac{PV}{kT} = \ln \mathcal{Z} = \sum_k \ln\left(1 + ze^{-\beta\varepsilon_k}\right) = \int_0^\infty g(\varepsilon)\ln\left(1 + ze^{-\beta\varepsilon}\right)d\varepsilon$$

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{z^{-1}e^{\beta\varepsilon_k} + 1} = \int_0^\infty g(\varepsilon)\frac{1}{z^{-1}e^{\beta\varepsilon} + 1}d\varepsilon$$

Also, use the Sommerfeld expansion for large $y = \frac{\mu}{k_B T}$.

$$f_n\left(e^y\right) = \frac{y^n}{\Gamma(n+1)}\left[1 + n(n-1)\frac{\pi^2}{6}\frac{1}{y^2} + n(n-1)(n-2)(n-3)\frac{7\pi^4}{360}\frac{1}{y^4} + \cdots\right]$$

Q.2.

Consider a free Fermi gas in two dimensions confined to a square area $A = L^2$.

(1) Calculate the Fermi energy.

(2) Derive the expression of the density of states.

(3) Derive the expression of the chemical potential as a function of T, A, N.

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Thermo & Stat Mech Hw #8

Q.1 $g(\varepsilon) = D \quad \varepsilon > 0$ total # of electrons is N

$g(\varepsilon) = 0 \quad \varepsilon < 0$

Given: $$\frac{PV}{kT} = \ln \mathcal{Z} = \sum_k \ln\left(1 + z e^{-\beta\varepsilon_k}\right) = \int_0^\infty g(\varepsilon)\ln\left(1 + z e^{-\beta\varepsilon}\right) d\varepsilon$$

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{z^{-1} e^{\beta\varepsilon_k} + 1} = \int_0^\infty g(\varepsilon) \frac{1}{z^{-1} e^{\beta\varepsilon} + 1} d\varepsilon$$

Sommerfeld expansion for large $y = \frac{\mu}{kT}$

$$f_n(e^y) = \frac{y^n}{\Gamma(n+1)}\left[1 + n(n-1)\frac{\pi^2}{6}\frac{1}{y^2} + n(n-1)(n-2)(n-3)\frac{7\pi^4}{360}\frac{1}{y^4} + \dots\right]$$

(1) Calculate the Fermi Energy. $\varepsilon_F = ?$

$$g(T,V,z) = \frac{PV}{kT} = \int_0^\infty D \ln\left(1 + z e^{-\beta\varepsilon}\right) d\varepsilon \rightarrow \text{Integration by parts}$$

$$N(T,V,z) = D\int_0^\infty \frac{1}{z^{-1} e^{\beta\varepsilon} + 1} d\varepsilon = \boxed{\frac{D f_1(z)}{\beta} = N}$$

$u = \ln(1 + z e^{-\beta\varepsilon}) \qquad v = \varepsilon$

$du = \frac{z e^{-\beta\varepsilon}(-\beta)}{1 + z e^{-\beta\varepsilon}} d\beta \qquad dv = d\varepsilon$

(14.26) $\rightarrow f_n(z) \approx \frac{(\ln z)^n}{n!}$

$$\int u\,dv = uv - \int v\,du$$

(14.41)

$$U = -\frac{\partial}{\partial\beta}\ln\mathcal{Z} = +\frac{D}{\beta^2} f_2(z)$$

$$= \frac{N}{f_1(z)} \cdot \frac{f_2(z)}{\beta}$$

$$\boxed{U = NkT\frac{f_2(z)}{f_1(z)}}$$

$U = \int_0^\infty d\varepsilon\, D\,\Theta(\mu - \varepsilon)\varepsilon = \int_0^\mu$

$U = \int_0^\mu D\varepsilon\, d\varepsilon \qquad N = D\mu$

$U = \frac{D\mu^2}{2} \qquad \frac{U}{N} = \frac{\mu}{2}$

$\boxed{\mu = \frac{N}{D}}$

$$\varepsilon \ln\left(1 + z e^{-\beta\varepsilon}\right)\Big|_0^\infty + D\int \frac{\beta\varepsilon z e^{-\beta\varepsilon}}{1 + z e^{-\beta\varepsilon}}$$

0

$$g = D\int \frac{\beta\varepsilon z e^{-\beta\varepsilon}}{1 + z e^{-\beta\varepsilon}} d\beta$$

$x = \beta\varepsilon \qquad dx = \beta d\varepsilon$

$$= D\int \frac{x z e^{-x}}{1 + z e^{-x}}\frac{dx}{\beta}$$

$$\boxed{g = \frac{D}{\beta} f_2(z)}$$

(2) Sommerfeld exp. for $f_1(z)$ and $f_2(z)$:

(2) should give same answer!

$$f_1(e^{\beta\mu}) = f_1(e^y) = \frac{y}{\Gamma(2)}\left[1 + 0 \dots 0\right]$$

$$= \frac{\mu}{kT}$$

$$f_2(e^y) = \frac{y^2}{\Gamma(3)}\left[1 + \frac{2\pi^2}{6}\frac{1}{y^2} + 0\right]$$

$$= \frac{y^2}{2} + \frac{\pi^2}{6y} = \frac{\mu}{2kT} + \frac{\pi^2 kT}{6\mu}$$

$(\mu = \varepsilon_F)$ b/c

From eq for N: $\frac{D}{\beta} f_1(z) = N \rightarrow \frac{D}{\beta}\frac{\mu}{kT} = N \rightarrow \mu = \frac{N}{D} \rightarrow \boxed{\varepsilon_F = \frac{N}{D}}$

(3) Rel. b/w pV & U ?

$$U = NkT\,\frac{f_2(z)}{f_1(z)} \qquad pV = kT\left(\frac{D}{\beta}\right) f_2(z)$$

AND $\dfrac{D}{\beta} = \dfrac{N}{f_1(z)}$

$$\boxed{U = pV}$$

(4) $U = \dfrac{NkT f_2(z)}{f_1(z)} = \dfrac{NkT \frac{\mu}{2(kT)^2}\left(1 + \frac{\pi^2}{6}\frac{kT}{\mu}\right)}{\frac{\mu}{kT}}$

$$U = \frac{N}{2}\left[\mu + \frac{\pi^2}{3}\frac{(kT)^2}{\mu}\right] \qquad \mu = \frac{N}{D}$$

$$C_V = \frac{\partial U}{\partial T} = \frac{\partial}{\partial T}\left(\frac{N}{2}\right)\left(\frac{N}{D}\right)\left[1 + \frac{\pi^2}{3}\left(\frac{(kT)^2 D^2}{N^2}\right)\right]$$

$$= \frac{\cancel{N^2}}{2\cancel{D}}\,\frac{\pi^2}{3}\cdot\frac{\cancel{D^2}}{\cancel{N^2}}\,\cancel{2}(kT)(k)$$

$$\boxed{C_V = \frac{1}{3}\pi^2 k^2 D T}$$

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Q.2) free Fermi gas → 2D $A = L^2$

1) Fermi energy $\varepsilon_F = ?$ $\varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2}\left(N/2\right)^2$

N particles
$\frac{N}{2}$ - highest energy state.

$\varepsilon_k = \frac{\hbar^2 \vec{k}^2}{2m} = \frac{\vec{p}^2}{2m}$ 2D

$\to p = \sqrt{2m\varepsilon}$
$dp = \frac{2m}{2\sqrt{2m\varepsilon}} d\varepsilon$

From (13.5)

$$\frac{U}{N} = \frac{\int d\varepsilon g(\varepsilon)\,\Theta(\mu-\varepsilon)\,\varepsilon}{\int d\varepsilon g(\varepsilon)\,\Theta(\mu-\varepsilon)}$$

$$\Sigma = \int \frac{d^2\vec{r}\, d^2\vec{p}}{h^2}$$

$$= \frac{2\pi L^2}{h^2}\int \sqrt{2m\varepsilon}\,\frac{m}{\sqrt{2m\varepsilon}}\, d\varepsilon$$

$$\boxed{\varepsilon_K = \frac{h^2 N}{2\pi g A m} = \mu} \ (3)$$

$$\Sigma = \frac{m 2\pi L^2}{h^2}\varepsilon$$

2) $\boxed{g(\varepsilon) = \frac{d\Sigma}{d\varepsilon} = g\frac{2\pi L^2 m}{h^2}}$ degeneracy

We can find μ from similar manner to part Q.1

$$N = \int d\varepsilon g(\varepsilon)\,\Theta(\mu-\varepsilon)$$

$$= \int d\varepsilon \frac{g 2\pi L^2 m}{h^2} \Rightarrow \frac{h^2 N}{2\pi g L^2 m} = \mu$$

A

3) $\therefore \boxed{\mu = \frac{h^2 N}{2\pi g A m}}$

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